

MONTHLY NOTICES
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MEETING OF 1939 JANUARY 13

Dr. H. Spencer Jones, President, in the Chair.

The election by the Council of the following Fellows was duly confirmed :—

- Wilfred Atkinson, 2 Duke Street, Penrith, Cumberland ;
Abdulazim Aziz, Department of Agriculture, Nicosia, Cyprus ;
Benjamin Bradley, B.Sc., 63 John Street, Brierley Hill, Staffs ;
Gustav Conradie, Lichfield, Stellenbosch, South Africa ;
Harold Walter Cox, 47 Upper Green, East, Mitcham, Surrey ;
Henry Rainsford Hulme, M.A., Ph.D., Royal Observatory, Greenwich,
London, S.E. 10 ;
James Gall Inglis, F.R.S.E., 36 Blacket Place, Edinburgh 9 ;
George Johnstone, Harbour Master, Braemar, Mostyn, Flintshire ;
Dhundiraj Govind Kale, B.A., 27 Garrick Avenue, London, N.W. 11 ;
Norman G. Matthew, Secretary, Astronomical Society of Edinburgh,
126 West Savile Terrace, Edinburgh 9 ;
Thomas Henry Locke Roberts, 105 Oxtou Road, Birkenhead ;
Adolph H. Rosenthal, 32 Kensington Park Gardens, London, W. 11 ;
Hermann von Socher, Übungs-Sternwarte der Universität Berlin,
Siegmundshof 21, Berlin, N.W. 87 ;
G. Temple, Ph.D., D.Sc., Professor of Mathematics, University of
London, King's College, Strand, London, W.C. 2 ;
William Cameron Walker, M.Sc., Ph.D., The Boulevard Secondary
School, Hull, Yorkshire ;
Harley Weston Wood, M.Sc., First Assistant Astronomer, Observatory,
Sydney, N.S.W., Australia.



The following candidates were proposed for election as Fellows of the Society, the name of the proposer from personal knowledge being appended :—

Lawrence Alfred Cox, 47 Upper Green, Mitcham, Surrey (proposed by R. L. Waterfield) ;

Henry Charles King, 238 Farnham Road, Slough, Bucks (proposed by W. H. Steavenson) ;

Frederick Jones O'Connor, M.A., Trinity College, Dublin (proposed by M. A. Ellison) ; and

George William Reed, M.Sc., Royal Observatory, Cape of Good Hope, S. Africa (proposed by J. Jackson).

Eighty-two presents were announced as having been received since the last meeting.

The President announced that the Gold Medal of the Society had been awarded by the Council to Monsieur Bernard Lyot, of the Meudon Observatory, for his observations and photography of the Solar Corona in the absence of a total solar eclipse.

The President further announced that the late Mr. Arthur Stanley Williams, for many years a Fellow of the Society, had bequeathed to the Society all his astronomical instruments, books, maps, photographs and manuscripts, together with a legacy of £200 for reduction or publication of his observations of variable stars ; and also a further legacy of £100 for the furtherance of astronomy in general or of the Society in particular as the Council of the Society may think fit.

Dr. R. Prager gave an account of the progress of *Geschichte und Literatur des Lichtwechsels der Veränderliche Sterne*, and of his recent work on the periods of variable stars.

A MODIFICATION OF THE MICHELSON INTERFEROMETER METHOD FOR THE DETERMINATION OF STELLAR DIAMETERS.

Reinhold Fürth, Kurt Sitte and Hans Peter Appel.

(Communicated by Sir A. S. Eddington)

1. *The Original Method of Michelson.*—The method for the measurement of diameters of fixed stars or the distances of double stars invented by Michelson * consists in sending the light, emitted from the object, by two different paths into the objective of the telescope. This can be realized in the most simple way by placing two parallel slits with micrometrically variable and measurable distance d in front of the objective. The image of the star appears striped because of the interference of the two beams of light, the visibility of the stripes varying with d . The interference phenomenon disappears for certain values of the distance.

The theory of the phenomenon was also given by Michelson. The most important results of it are the following: Let us first suppose the light source to be a slit of the width a at a distance X from the observer, emitting monochromatic light of wave-length λ . Then the apparent width α of the slit is $\alpha = a/X$. Further, let I_1 be the maximum and I_2 the minimum of the intensity inside the interference image. We define as “visibility” of the interference phenomenon the expression $V = \frac{I_1 - I_2}{I_1 + I_2}$. Putting finally $\alpha_0 = \lambda/d$, we get

$$V = \frac{\sin \pi \frac{\alpha}{\alpha_0}}{\pi \frac{\alpha}{\alpha_0}}. \quad (1)$$

It is obvious that V varies periodically with α . For $V = 0$ the interference fringes disappear totally, which according to (1) is the case, if α is an integral multiple of α_0 , i.e. if the equation

$$\alpha = n\alpha_0, \quad (2)$$

with $n = 1, 2, 3, \dots$ is satisfied.

If V is measured for any distance of the interferometer slits, the required α can be calculated from (1). The easiest procedure consists in determining those values of d , for which the interference fringes disappear. Then we get α from equation (2), where n means the “order” of vanishing, i.e. the number of disappearances which occur during the increase of d up to the value concerned.

Quite similar results are obtained if the light-source is a star with the

* A. A. Michelson, *Ap. J.*, 47, 283, 1918.

apparent diameter δ . Instead of equation (1) a corresponding equation has been derived by Michelson, which is shown in fig. 1 together with the function (1) (the fully drawn curve represents V for a slit, according to (1); the dotted

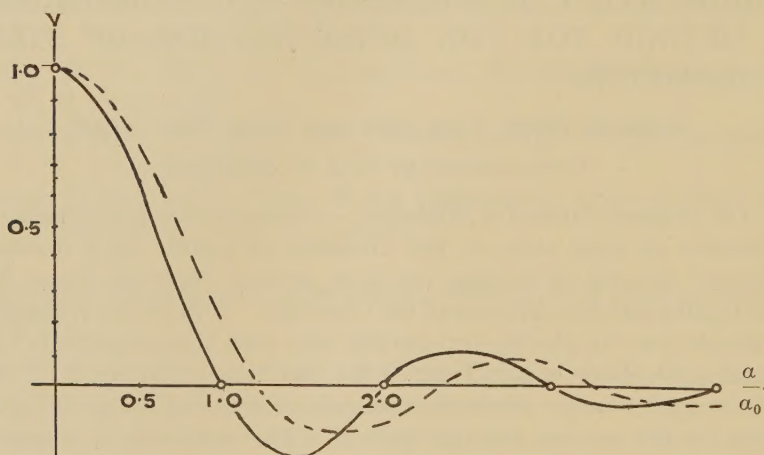


FIG. 1.

curve represents V for a star). The interference fringes disappear in this case if, instead of equation (2), the equation

$$\delta = 1.22 a_0 \cdot n \quad (3)$$

is satisfied. For double stars with a separation δ' the vanishing of the interference fringes is determined by the equation

$$\delta' = 0.5 a_0 \cdot n. \quad (4)$$

According to what has been said, the application of the Michelson method to the measurement of stellar diameters seems to be very easy. But its realization meets with great difficulties, as the image of the star in the telescope generally moves very quickly because of the air perturbations. It is therefore impossible to observe the disappearance of the interference fringes taking place in certain positions of the interferometer or to determine their visibility photometrically. For the same reason it is impossible to use a photographic method instead of the visual, as the intensity of light is not sufficient for short exposure times, and long exposure would cause a blurring of the photograph. Therefore exact measurement with the Michelson method has been possible as yet only in the rare case of extremely small air perturbations.

If one succeeded in eliminating the irregular movement of the interference image by a suitable optical arrangement, the application of the Michelson method to astrophysics would become much easier and the measurement of very feeble light-sources would also be possible, as the interference phenomenon could be photographed with exposure times of arbitrary lengths.

The present problem was pointed out by Prof. E. F. Freundlich in one of a course of lectures delivered in the winter 1936-37. The idea of the modified method described below was conceived by one of us (Fürth) as a result of considering this problem. The main feature and the simple theory of the method was developed by Fürth and Sitte on the basis of the positive results of preliminary experiments. Therefore it seemed useful to develop the method by further systematic experiments in the laboratory, so that the astronomical applications would meet with no principal difficulties. These experiments were carried out as a dissertation by H. P. Appel under the direction of Fürth and Sitte in the Physical Laboratory of the German University in Prague and are also described below. We are very indebted to Prof. Freundlich for numerous interesting discussions. We thank the Bohemian Astronomical Society for lending us the telescope objective, and we thank the representatives of the firm Leitz-Wetzlar, and Prof. A. Pascher of the German University in Prague for lending us the "Leica-Mikro-photoansatz."

2. *The Principal Ideas of the Modified Method.*—To explain the principal ideas of the modified method, we start from a model where the real image of the star with interference fringes formed by the objective of the telescope is replaced by a grating with narrow lines illuminated by a parallel beam of light. If this grating is quickly and irregularly displaced in its plane, its structure cannot be discerned visually or photographically, if the intensity of light is very small. If, however, the grating is observed by a telescope focused to infinite distance, the diffraction phenomenon caused by the grating appears as a system of interference fringes, which remain at rest in spite of the movement of the grating. This phenomenon can therefore be observed and photographed without any difficulty and the existence and structure of the grating may be deduced from it in an indirect way. The "objective-grating method," well known and frequently used in astrophysics, depends on the same fact: that only the grating element, and not the position of the grating, is decisive for the existence and position of the diffraction image.

If there is good contrast, the system of the Michelson interference fringes is nothing else than a system of strips emitting coherent light just like a grating (the differences will be pointed out later). It must therefore be possible to produce interference between the light beams proceeding from the light fringes by the same optical arrangement as in the case of a grating. So it is to be expected that the resulting "secondary" interference fringes will not move with the "primary" Michelson fringes and that these secondary interferences will be observable even on nights with strong air-perturbations.

Furthermore, it is clear that the distinctness of the secondary interferences must change proportionately to that of the primary: of course, no secondary interferences can be formed if the primary Michelson system of fringes disappears at a suitable distance of the slits; and if the contrast of the primary interferences has its maximum, the secondary will show its optimal visibility. Therefore apparent stellar diameters can be derived from the visibility of the secondary interferences instead of that of the primary.

As we know the distribution of the light intensity in the primary interference image, it would not be difficult to deduce from it the distribution in the secondary interferences. But this is not necessary because it is sufficient, as we have seen already, to measure the distance d of the slits, for which the intensity contrasts of the primary interferences disappear. For this same value of d also the secondary interferences become imperceptible.

Up to this point we have simplified the problem by the supposition that the system of primary interferences is equivalent to a normal grating. But this is not quite correct; it follows from Michelson's theory that the distance between the successive maxima are not equal. Thus we have to replace the interference image by a grating with a "variable grating element." But

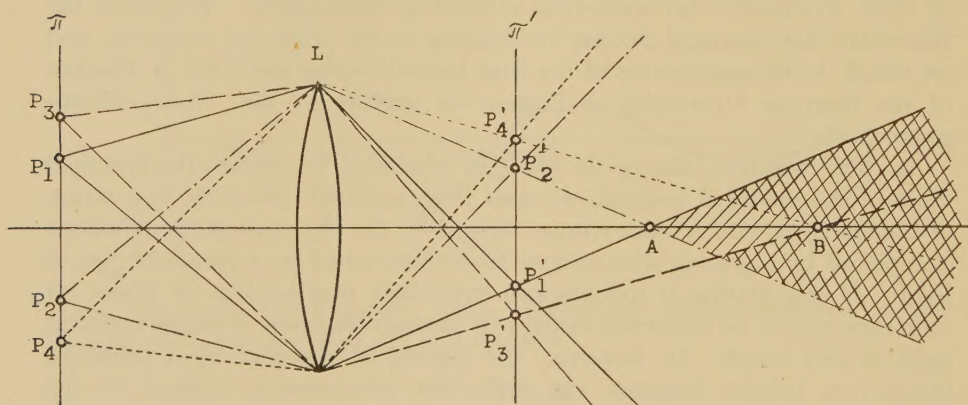


FIG. 2.

the usual arrangement with a grating of such a kind would give no maxima and minima of intensity. Consequently a telescope placed behind the Michelson interference image and focused to infinite distance would not show any "secondary" interference phenomenon.

We succeed nevertheless on account of the fact that the distances between maxima of low orders in the primary Michelson interference image are sufficiently constant. If we eliminate the effect of the maxima of high orders we may get systems of fringes, which are not perfectly distinct, but still easily discernible. This can be attained by a simple optical trick explained by fig. 2.

We use a model of four points P_1 , P_2 , P_3 and P_4 , emitting coherent light and situated in a plane π symmetrically to the optical axis of the lens L . Their real images P'_1 , P'_2 , P'_3 and P'_4 are formed in the plane π' . The beams emerging from the points P_1 and P_2 overlap in the hatched region, which begins with the point A . If there were only these two radiating points, interference fringes would be observable in the whole hatched region. But they are destroyed again by the interference effect of the two outer light-points in the whole region, into which penetrate the light-cones of P_3 and P_4 , together with those of P_1 and P_2 ; this region is marked by double-hatching. No clear differences of light intensity will be observable in this region because of the superposition of the different systems of interference.

In the single-hatched region between A and B, however, clear interference images must be observable.

Now in our case of a great number of not exactly equidistant light fringes there must exist behind a collecting lens also a region, inside of which only the fringes in the middle of the primary Michelson image with approximately equal distances are effective. In this region we shall find our secondary interferences. At a larger distance from the lens, however, the secondary interferences lose their distinctness and grow more and more indistinct because of the action of the outer primary fringes, their distances being considerably different from those of the inner fringes. The interference phenomenon observable in the region mentioned above will undergo only small variations, while the primary image is moving. To prove this, the following laboratory experiments were made.

3. *The Experimental Arrangement.*—The experiments were carried out by using an "artificial star" in order to reduce considerably the exposure

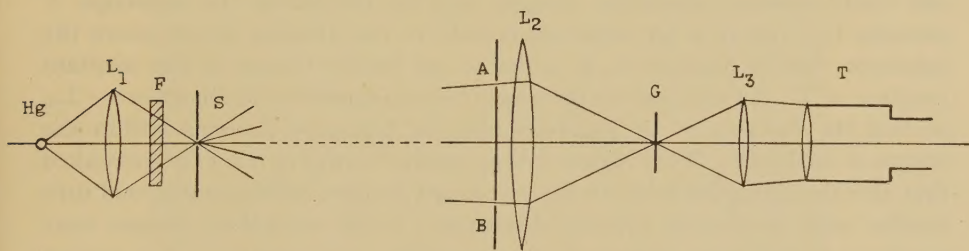


FIG. 3.

times for the photographic registration. The "star" consisted either of a small circular aperture or of a narrow slit (S) in a screen on which the light of a mercury-arc lamp (Hg) was concentrated by a condenser lens (L₁) after having passed through a filter (F) transparent only for the green light of the spectral line 5460 Å. A diagram of the optical arrangement is shown by fig. 3. At a distance of about 14.5 m. from the star a Zeiss U.V. objective lens (L₂), with a diameter of 21 cm. and a focal length of 340 cm., was placed. In front of the objective a screen was fixed consisting of two brass plates sliding in rails. Each of these plates contained a slit of 15 cm. length, the width of which could be varied between 0.5 and 1.0 cm. The plates were joined by a folded black paper, so that light could fall into the objective only through the slits (A and B). The distance between the slits was read off on a scale engraved into the rails. In order to allow visual observation of the real image formed by the objective a ground glass (G) could be placed there (*i.e.* at a distance of 4.43 m. from the lens). The observation or photographic registration of the secondary interferences was carried out by means of the lens L₃ and the telescope with photographic equipment T. The focal distance of L₃ was 18 cm., that of the telescope objective 34 cm. The distance between the primary interference image and the telescope T was kept constant (54 cm.); the lens L₃ had to be moved in order to get observable secondary fringes, when the distances of the primary fringes were varied.

The real images of the primary fringes were in all cases situated inside the telescope T, not far from its objective. They also could be observed visually or photographed, if L_3 was subjected to a suitable displacement. Thus secondary interferences were produced just in the same way as represented in fig. 2 for a simplified case.

Two experimental facts demonstrate unambiguously the correctness of our considerations concerning the formation of the secondary interferences. When the distances of the maxima of the primary fringes were diminished (*e.g.* by varying the dimensions of the light-source) it was, firstly, necessary to displace L_3 towards G in order to get distinct secondary fringes, and besides one observed a diminution of the distances of the secondary fringes. These features are immediately shown by fig. 2. We see that a diminution of the distances between the points P_1, P_2, P_3 and P_4 in π corresponds to a proportional decrease of the distances in π' , and that, therefore, the region of visibility would be closer to π' and smaller. The position of L_3 fixed, one could observe secondary fringes only by displacing the telescope T towards L_3 ; but it is far more convenient in the practice not to move the telescope, but to displace L_3 in order to get visible fringes at the constant position of T. In this case we obviously have to diminish the distance $\pi - L_3$, so that the distance $\pi - \pi'$ increases and the telescope remains within the region of visibility. This region will be smaller now, but it is to be remarked that also the distances between the secondary fringes, consequently, will turn smaller with decreasing primary distances. If the secondary fringes were produced in the same manner as the interference phenomenon in a grating spectrograph, the contrary would occur: a diminution of the distances of the primary fringes would correspond to a diminution of the grating element and therefore an increase of the diffraction angles would result.

In order to test the experimental arrangement, firstly, a few determinations of the width of the slit S and of the diameter of our artificial star were made. We will give here only the results of a series carried out with the slit. In these previous experiments we did not change the distance of the interferometer slits A and B, but the width of the light-emitting slit S, which comes to the same thing. Thus we keep constant the value of $a_0 = \lambda/d$ and change that of $a = a/X$; on account of equation (2) we must expect to find a series of widths a , for which the contrasts will disappear. So we made the measurements with a constant d (2.5 cm.), gradually enlarging the width of the slit S and noting the values a , for which no primary fringes could be perceived. We attribute the orders 1, 2, 3, . . . respectively to them and calculate for each "effective" the corresponding "theoretical" width according to equation (2). The effective widths were measured directly by means of a microscope. Table I contains the results.

With the exception of the single value at $n=2$, which may be considered as disturbed, all the deviations are of an absolutely tolerable order of magnitude. The fact that all the a' are larger than the corresponding a is not so surprising as it seems to be at first sight. It is a consequence of the procedure we adopted in making the measurements: we enlarged the slit gradually and read off its width a , when the contrasts just disappeared; thus that a is

TABLE I

Effective Width a (in mm.)	Order	Theoretical Width a' (in mm.)	Deviation (in per cent.)
0.380	1	0.396	4
0.665	2	0.792	16
1.110	3	1.180	5.9
1.460	4	1.580	7.6
1.880	5	1.980	5

the lower limit of an interval of values, inside of which the calculated a' must be. If, on the contrary, we had diminished the width in the course of a series, the directly measured a would have been the upper limit of this interval. This assertion was also experimentally verified.

Further, it was necessary for the proper purpose of our investigation to produce an "artificial air perturbation." We realized it by placing a narrow

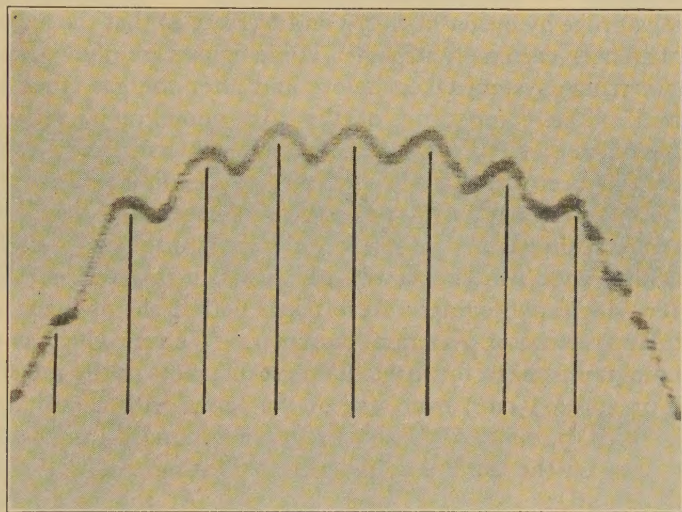


FIG. 4.

gas stove of about 25 cm. length below the path of the rays close to the interferometer slits, taking care that no light from the flames could get into the objective. By this device such strong air perturbations could be produced during the experiments as certainly never occur at astronomical measurements. We shall demonstrate this by means of two photographs in the last paragraph.

The photographs were taken with a Leica-camera in connection with the microphotographic arrangement to the "Leica"-camera, the "Mikas" of the firm Leitz-Wetzlar, on Leica-film "Agfa-Pankino," because this film has a high sensibility for green light. Nevertheless exposure times of 7 to 9 minutes had to be used.

Finally we will give evidence for the correctness of our assertion that at least a sufficient number of the primary Michelson fringes may be considered as a system of practically equidistant radiating lines. For this purpose we

made a microphotometric registration of a photograph of a primary interference image, which is reproduced in fig. 4. The maxima of the intensity are marked by vertical lines, which are in fact equidistant to a sufficient approximation. Thus all our suppositions are proved to be correct and the arrangement is demonstrated to be efficient.

4. *The Results.*—The definite measurements were made with a width of the interferometer slits of 0.5 cm., instead of 1.0 cm., as in this case the number of the primary fringes was about 30 per cent. larger, and besides the maxima and minima became more distinct. Consequently the secondary fringes became more numerous (up to 50 per cent.) and more distinct too. Further, we made these decisive experiments with a constant width of the radiating slit, varying the distinctness of the interference system by displacing the interferometer slits, *i.e.* by varying d , as in real astronomical measurements. Each series consisted of 8 to 15 photographs taken for various distances d between 1.8 and 10 cm.

We, firstly, prove by means of figs. 5 and 6 (Plate 4) that the air perturbations really had been extremely strong. These figures are photographs of the same primary fringe system taken before and after the gas stove had been put in action. The comparison of the two photographs shows clearly that, in consequence of the displacement of the interference image caused by the strong air perturbation, the contrasts have disappeared completely and the total breadth of the bright spot has increased.

We have taken altogether five series of photographs, using as light-source a slit 0.152 mm. wide in the first two series and 0.106 mm. in the others. Besides a few single photographs were taken with a slit of 0.302 mm. and with a circular aperture of 0.32 mm. diameter. We reproduce as examples only the photographs taken for three different distances of the interferometer slits (1.8, 3.05 and 7.1 cm.) from one of the series with a radiating slit of 0.106 mm. Figs. 7, 10 and 13 show the corresponding primary fringes, figs. 8, 11 and 14 the secondary fringes at resting air, and figs. 9, 12 and 15 the secondary fringes at disturbed air. At the first distance the images are perfectly distinct, at the second distance they have become indistinct, and at the third distance the fringes have disappeared entirely; further, we see the complete parallelism in the behaviour of the primary and the secondary interferences, which is a fundamental supposition of our method. Finally it is proved that the air perturbations do not destroy the distinctness of the secondary fringes; they remain practically at rest, as we supposed.

Table II contains the numerical results of the five series mentioned above, obtained for $n = 1$, as the disappearance of the interferences in this case could be defined exactly only for the first order. Again, as in Table I, we find the "calculated" values d' smaller than the corresponding "observed" d , because these latter values were also determined as the lower limits of the intervals, inside of which the interference phenomenon disappears.

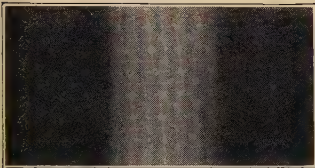
Comparing the Tables I and II, we see that our arrangement permitted the determination of visual angles with an uncertainty of about 4–8 per cent. by using the original Michelson primary fringes, and that the uncertainty



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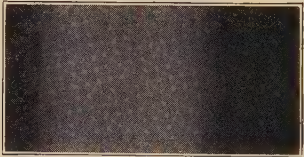
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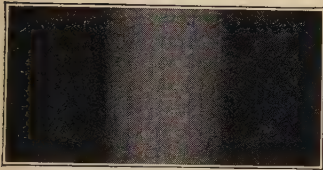
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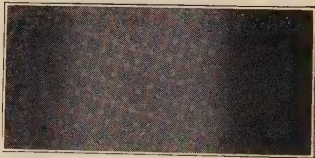
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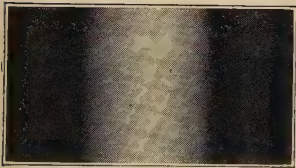
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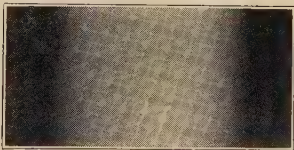
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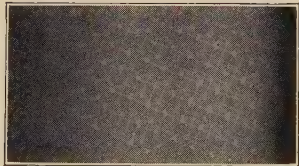
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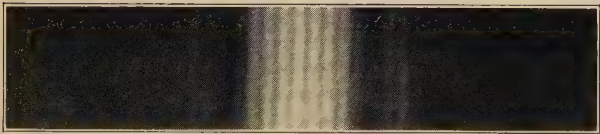
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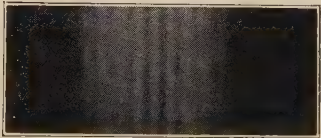
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*Reinhold Fürth, Kurt Sitte and Hans Peter Appel,
A Modification of the Michelson Interferometer Method for the Determination of Stellar Diameters.*

TABLE II

"Calculated" Distances d' (in cm.)	"Observed" Distances d (in cm.)	Deviation (in per cent.)
4.37	3.80	12.2
4.37	4.05	7.3
7.55	6.55	13.5
7.55	7.05	6.0
7.55	7.05	6.0

increased to 6–14 per cent., *i.e.* to about the double amount, by working with the secondary fringes, which is a rather satisfactory result. Moreover, it is to be considered that the experimental conditions were made on purpose much more unfavourable than those appearing in astronomical practice.

Finally we show in figs. 16 and 17 two photographs of secondary fringes in still and disturbed air, produced by an "artificial star" with a diameter of 0.32 mm. Again we see that the air perturbations practically have no influence upon the clearness of the interference figures.

In our opinion these laboratory experiments have given evidence that the Michelson interferometer method, completed by a simple supplementary arrangement, can be successfully used for the measurement of stellar diameters even if there is strong air perturbation. The uncertainty of the modified method is somewhat larger, but its application will not be limited to the rare condition of extremely good weather.

Summary.—It is known that the Michelson interferometer method for the determination of stellar diameters can be used only in the rare cases of small air perturbations, as otherwise the image of the star in the telescope moves so quickly that the observer is not able to perceive its interference fringes. Proceeding from the consideration that the system of fringes produced by a diffraction grating does not move, when the grating is translated in its plane, it is to be expected that by a suitable optical arrangement a "secondary" image of the "primary" Michelson interference image can be produced, which remains in rest even when the primary image is moving because of the air perturbations. The correctness of this reasoning was proved by laboratory experiments. By the new method interferometrical measurements of objects can be made with an accuracy of 10 to 15 per cent. even if there are strong perturbations. It is therefore to be expected that this new method can be successfully used for the measurement of stellar diameters.

*Physical Institute,
German University,
Prague.*

STARK EFFECT IN B STARS.

J. Stuart Foster, F.R.S., and A. Vibert Douglas, M.B.E.

I. INTRODUCTORY DISCUSSION OF STARK EFFECT

Stark effects in certain B stars are revealed not only by a great broadening of the Balmer lines which remains unexplained on any other basis, but by the appearance of many electric combination lines in the absorption spectrum of neutral helium. Although the interesting orthohelium line $\lambda 4470$ ($2p-4f$), most frequently cited as evidence for a stellar Stark effect, actually was found by Struve * with a wave-length and intensity neither of which appears to be due to the influence of electric fields alone, the variety of new evidence from the helium spectrum given below is in somewhat better agreement with simple theoretical expectations, and removes all doubt whatever as to the reality of Stark effects in stellar atmospheres.

Before considering this evidence in detail the main facts concerning Stark effects in helium will be reviewed briefly in order to bring out their interesting complexity, and especially to correct certain representations of the effect which consist in part of an unjustified enlargement upon an early theoretical interpretation.

While the Stark displacements in hydrogen are rather accurately proportional to the external field strength (except in very low fields where effects associated with the fine structure of the lines enter) † it is not possible to give any corresponding simple description of the effects observed in helium or in any other spectrum. By an application of quantum mechanics ‡ it was learned that this circumstance is directly related to the fact that the separations of the helium sub-levels characterised by different values for the quantum number l are large compared with the corresponding separations within the fine structure of the hydrogen level of the same principal quantum number. Since in any case the displacements will become linear with the field only after they are large compared with the above mentioned zero field separations, the result is that one sees in helium under the average experimental range of fields a complexity of displacements and intensities which at maximum field may or may not be smoothened out into a nearly hydrogen-like regularity. That will partly depend upon the zero field separations of the initial levels involved in any line group under consideration as well as the associated principal quantum number, which factors may be said to control in the main not only the related displacements within the group but also through them the intensities. Consequently in

* O. Struve, *Ap. J.*, **74**, 4, 247, 1931.

† Schlapp, *Proc. Roy. Soc.*, A, **119**, 313, 1929.

‡ J. S. Foster, *Proc. Roy. Soc.*, A, **117**, 137, 1927.

reviewing the entire helium spectrum one may expect to find at any given field strength results as varied as are these factors.

Speaking more specifically, all sharp series lines ($2p\text{-}ns$; $2P\text{-}nS$) are displaced toward the red without exception by an amount which is usually small because the s or S term has the greatest separation from other terms. As in hydrogen, the displacement increases rapidly with n , and indeed to such an extent that some sharp lines examined in this work show a decided broadening toward the red.

The principal series lines ($2s\text{-}np$; $2S\text{-}nP$) are of small practical value in this astrophysical research because the few lines which penetrate the atmosphere have no pronounced Stark effect (except parhelium λ 3965, masked by H_{ϵ} λ 3970), and the values of the final terms ($2s$, $2S$) are such that later members of these series with appreciable Stark effects are thrown into the ultra-violet region. The orthohelium lines are shifted toward the red, while members of the corresponding parhelium series are displaced toward the violet. This contrast is a consequence of the inverted P terms of parhelium.

Diffuse series lines are of importance in any investigation of Stark effects. The lines are strong; and the shifts, which are larger than in the sharp or principal series, are toward the red. An exception with regard to the sense of the displacement will be found in the first member of each series ($\lambda\lambda$ 5876, 6678) since here there is no close f (or F) term to repel the diffuse term. For similar reasons, this exception applies generally to all first members of succeeding series of types $\left. \begin{smallmatrix} 2p \\ 2s \end{smallmatrix} \right\} - mf$; $\left. \begin{smallmatrix} 2p \\ 2s \end{smallmatrix} \right\} - mg$; $\left. \begin{smallmatrix} 2P \\ 2S \end{smallmatrix} \right\} - mF$; $\left. \begin{smallmatrix} 2P \\ 2S \end{smallmatrix} \right\} - mG$, etc. The shift toward the violet may persist through two or even several members of an electric combination series.

The stronger electric combination lines are usually so close to the corresponding diffuse line that complete resolution in stellar spectra is not possible. The details in each structure of interest will be given later.

We shall now make a few corrections to statements regarding Stark effects which are commonly quoted. (i) High fields are not necessary for Stark effects since 200 v./cm. is enough in either hydrogen or helium to produce observable effects in the laboratory. (ii) The relation of displacement to field strength is not linear in hydrogen and quadratic elsewhere. The linear relation develops in hydrogen at extremely low fields and persists over the observed range with only small second order effects superposed; in helium one finds in the ordinary range of fields the quadratic relation holding at least approximately for most sharp series lines, and for visible members of the principal series while the linear relation is developed at moderate fields for most diffuse series lines. The law applying to the majority of electric combination lines with which we shall have to deal is too complex to permit brief description. (iii) No members of fine structures or individual lines in either hydrogen or helium have ever been found with *symmetrical splitting* in Stark effect. Such marked group symmetry as is often observed is built up from individual line patterns which as a rule show

pronounced asymmetry. Thus in the Stark effect for a member of a diffuse series in stellar spectra one may expect a broadening due to absorption; but (aside from limitations already cited) effects due to external electric fields alone may be expected to lead only to red shifts.

II. OBSERVATIONAL MATERIAL

The literature upon Stark effect in stellar spectra prior to 1932 had opened up an attractive and important field of inquiry and had suggested the need of a very thorough investigation of all the evidence possible. Through the kindness of the Director, Dr. J. S. Plaskett, F.R.S., the writers were given the privilege of obtaining spectrograms with the 3-prism spectrograph attached to the 72-in. reflector of the Dominion Astrophysical Observatory, Victoria, B.C.* These were calibrated by means of the wedge in regular use there in 1932. Subsequently Dr. C. S. Beals and Dr. F. S. Hogg kindly took for us several spectrograms in the red and yellow regions. Microphotometer tracings have been made and the intensity graphs or line profiles have been constructed. The stars selected for study are:

(1)	88 γ Peg	B2	2 ^m .9	H.D. 886	16 plates.
(2)	85 ι Herc	B3	3 ^m .8	H.D. 160762	6 plates.
(3)	17 ζ Cas	B3	3 ^m .8	H.D. 3360	1 plate.
(4)	22 ζ Drac	B5	3 ^m .1	H.D. 155763	4 plates.

From this material 12 normal helium lines have been examined, 9 electric combination lines have been definitely identified, and the presence of five others found to be not incompatible with the observed profiles. Due to low intensities and large displacements these latter lines can only be expected to appear as weak band-like depressions of the continuous background. Comparison has been made with laboratory data both as regards displacement at different field strengths and relative intensities at these fields. A general summary of the theoretical behaviour and of the observations follows.

III. SUMMARY OF THE EVIDENCE FROM THE LINES

Wave-lengths indicate zero field positions and the signs + - indicate the direction of displacement with fields

λ	Notation	Theoretical Description	Actual Appearance in Stellar Spectra
3820 (+)	2p-6d	Electric fields displace it to red, intensity decreasing with field. Electric combination lines move to violet, intensity increasing with field.	Winged to violet, maximum spread of 10 A. corresponds to 25 Kv./cm.

* The writers are indebted to the National Research Councils of the U.S.A. and Canada for grants towards the expenses incurred in going to Victoria, B.C.

λ	Notation	Theoretical Description	Actual Appearance in Stellar Spectra
4009 (+)	$2P-7D$	Weak line; fields displace it to red.	Weak line; red wing affected by fields up to 8 Kv./cm.
4007 (-)	$2P-7P$	Low fields rapidly spread components to violet.	Definitely present in γ Peg for low field positions. (Regarded doubtful by Struve, <i>Ap. J.</i> , 1931 Nov., p. 247.) At higher fields the large displacement causes it to be lost in background.
4026.2 (+)	$2p-5d$	Fields displace to red. See fig. 3.	Winged on red side to λ 4030, indicating fields to 40 Kv./cm.
4025 ($\pm$)	$2p-5f$	Fields to 40 Kv./cm. move line to red 0.2 A. and back beyond zero field position to 0.2 A. to violet.	Only on assumption of the presence of this line with intensities as indicated in fig. 3 can the strong violet wing of the general profile near the zero position of d be explained.
4025 (-)	$2p-5g$	Fields displace to violet.	Extension of violet wing 4 A. indicates presence of this line with fields up to 40 Kv./cm. Lines f and g fully account for smoothed violet profile with no further hypotheses required.
4121 (+)	$2p-5s$	Fields displace to red.	Asymmetrical, winged to red. See fig. 5.
4143.77 (+)	$2P-6D$	Fields displace components to red. Shift 3 A. at 12 Kv./cm.	Red wing extends to 3 A.
4143 (+ +)	$2P-6F$	Components cross zero position of D and move 2 A. to red with fields to 20 Kv./cm.	Cannot be separately identified as it merges with D .
4143 (-)	$2P-6G$	Weak components. Maximum displacement 1.6 A. to violet at 20 Kv./cm.	Lines G and H merge to augment inner violet wing of general profile. They cannot be separately identified.
4143 (- -)	$2P-6H$	Weak components, displacement 4 A. to violet at 20 Kv./cm.	
4142 (- -)	$2P-6P$	Moves rapidly to violet and becomes more intense with increasing field.	Violet wing of general profile must be largely due to this PP line. In ζ Drac and ι Herc it is definitely observed in fields up to 15 Kv./cm.; in γ Peg a line at λ 4137 [recorded as unidentified by Struve (<i>Ap. J.</i> , 1931 Nov., p. 231)] could be this PP line at fields about 20 Kv./cm. But red wing becomes lost in continuous background at lesser fields.

λ	Notation	Theoretical Description	Actual Appearance in Stellar Spectra
4168.97 (+)	$2P-6S$	Fields displace to red.	Very asymmetrical. Red wing extends 2 A. indicating field 25 Kv./cm. See fig. 5.
4387.93 (+)	$2P-5D$	Fields displace to red. See fig. 4.	Red wing of profile extends to 40 Kv./cm. position.
4387 ($\pm$)	$2P-5F$	Displaced to red and back to violet ± 0.5 A. in fields to 40 Kv./cm.	Heavy absorption shown by strong inner portion of violet wing of general profile can only be explained in terms of this line.
4386 (-)	$2P-5G$	Displaced to violet 4 A. in fields to 40 Kv./cm.	Lines F , G , P provide a consistent and sufficient explanation of the entire violet wing of the general profile.
4383 (-)	$2P-5P$	Displaced to violet 3 A. in fields up to 30 Kv./cm.	
4438 (+)	$2P-5S$	Fields displace to red.	Weak asymmetrical line, winged to red.
4471.56 (+)	$2p-4d$	Fields displace to red 5 A. at 100 Kv./cm. See fig. 1.	Red wing of profile for stars under consideration indicates fields up to 60 to 80 Kv./cm.
4470.16 (- -)	$2p-4f$	Fields displace rapidly to violet.	Wide weak absorption from λ 4470 to λ 4466 is thus explained; but <i>not</i> the relatively strong line at λ 4471-4469 which has been almost invariably so identified.
4517.65 (-)	$2p-4p$	Fields displace slightly to violet.	Too weak to identify certainly in ϵ Herc; possibly present in γ Peg blending with weak line λ 4518.06 recorded by Struve (<i>Ap. J.</i> , 1931 Nov.) and identified with very weak line N III 4518-18.
4713 (+)	$2p-4s$	Fields displace to red.	Weak, asymmetrical, winged to red.
4921.93 (+)	$2P-4D$	Fields displace to red 3.3 A. at 40 Kv./cm. See fig. 2.	Red wing is explained by fields up to 50 Kv./cm.
4920.6 (-)	$2P-4F$	Fields displace to violet 2 A. at 40 Kv./cm.	Part of weak absorption of outer violet wing of general profile is thus fully explained, but <i>not</i> the relatively intense absorption near zero field position.
4911 (-)	$2P-4P$	Fields displace to violet.	Weak; moves to violet 1.7 A., indicating fields up to 40 Kv./cm.
5876 (-)	$3p-3d$	Strong line, very small shift to violet in extremely high fields.	No Stark effect observed.

IV. FIELD STRENGTHS

With the aid of laboratory data, it is possible to estimate from the observed profiles of both helium and hydrogen lines the *upper limits* of the fields that are appreciably affecting the photographic record.

Since the natural width of the lines has not been considered in this determination, some of the values are probably a little high. The earlier members of a series will obviously be expected to give evidence of the presence of higher fields.

Table of Field Strengths

As indicated by helium and hydrogen lines

Line	λ	γ Peg B2	ι Herc B3	ζ Cas B3	ζ Dra B5
(a) He (weak sharp)	4168	25 Kv./cm.	25 Kv./cm.	15 Kv./cm.	20 Kv./cm.
(b) He (diffuse ; red wings)	4922	50	...	...	...
	4471	70	70	80	70
	4388	40	40	25	20
	4026	40	35	30	30
	4144	15	15	15	10
	4009	8	8	...	...
	3820	...	25	...	...
(c) He (electric combination)	2P-4F	50	...	...	...
	2P-4P	40	...	...	...
	2p-4f	60	60	...	...
	2P-5F	40	40	...	...
	2P-5G	40	40	...	...
	2P-5P	30	30	...	...
	2p-5f	40	40	...	...
	2p-5g	40	35	...	...
(d) H (Balmer)	4861	150-200	150-200	...	180
	4341	45-60	65	75	75
	4101	35-50	40-50	35	40

V. INTENSITY MEASUREMENTS

1. The upper portion of fig. 1 is the theoretical displacement and intensity chart for 2p-4d and the accompanying electric combination line 2p-4f.* An important point which must be very strongly emphasized is that at its zero field position the f line has *intensity zero*, while its 10 Kv./cm. position is already 0.34 A. to the violet, and of intensity only one-eighth that of the d line at the same field. It thus seems unreasonable to attribute the heavy absorption at and near 4470, as shown in the profiles in fig. 1, to the f line.

A quantitative examination of these profiles will now be outlined.

* J. S. Foster, *Phys. Rev.*, 23, 6, 1924 ; *Proc. Roy. Soc.*, A, 114, 1927, and A, 117, 1927.

2. $\lambda 4471$, $2p-4d$. The peak of this line shows no appreciable shift to the red. Therefore a very large proportion of the atoms must be absorbing, if not actually at zero field conditions, then extremely close to zero, *i.e.* small fields predominate. The inevitable deduction from the spectrum

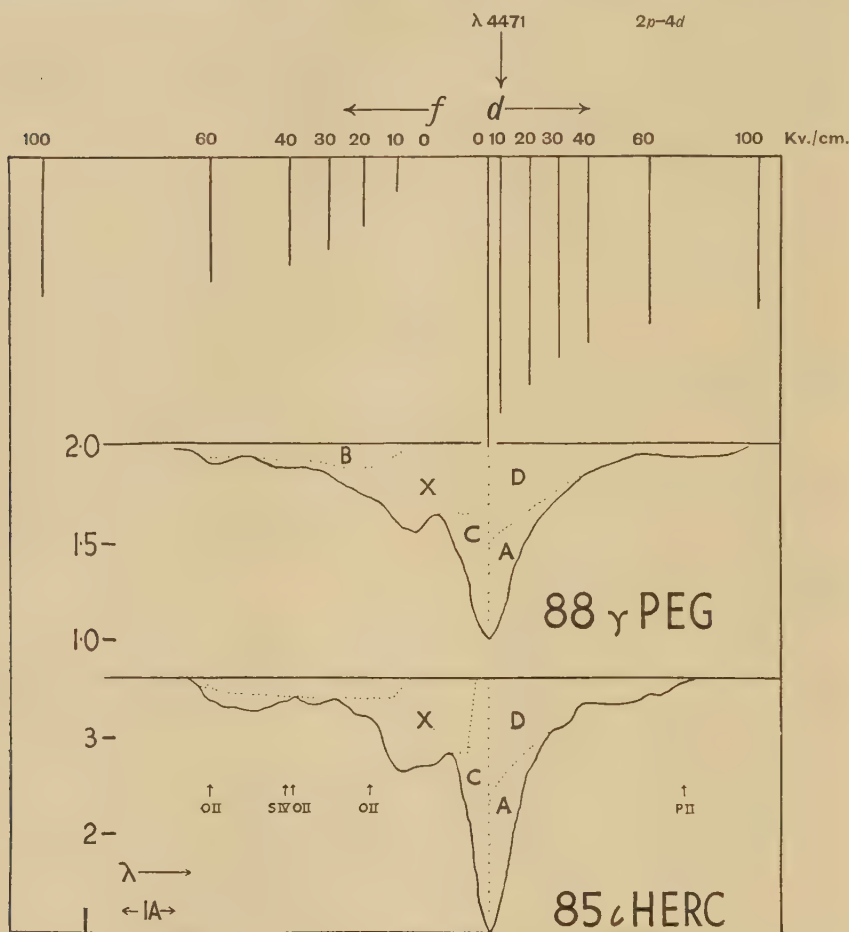


FIG. 1.

is in general agreement with the theoretical distribution of fields in a hydrogen atmosphere obtained by Verweij,* who, starting with zero number of atoms at zero field, finds that the ascent of the distribution curve is very rapid.

As a preliminary assumption we take 50 per cent. of the central ordinate as indicating the contribution to the line made by atoms at fields so close to zero that the term zero field is used in this connection throughout the present paper. This will be practically symmetrical to violet and to red where it can be represented by the area A of the red wing. The remaining area D is the contribution of the atoms being acted upon by appreciable electric fields up to 60 Kv./cm. or more. The ordinates at 10, 20, 30, 40

* S. Verweij, *Pub. As. Inst. U. Amsterdam*, No. 5, 1936.

and 60 Kv./cm. can be measured. If one uses the relative intensities of f and d lines as shown in the chart to find the corresponding f ordinates the resultant values can be indicated at the corresponding positions of the displaced f line. This method alone, however, may be misleading as it fails to take account of the differing amounts of displacement of the d and f lines. Hence the areas of the absorption corresponding to the displacements for field intervals 0-15, 15-25, 25-35, 35-50, 50-70 Kv./cm. are measured in arbitrary units and these areas are multiplied by the ratio f/d for a mid-position of each range, viz. 10, 20, 30, 40, 60 Kv./cm. (see Tables (a), (b)). The resulting areas must then be compared with the available area of the violet wing from the f (zero field) position to the f (60 Kv./cm.) position. A further refinement takes count of the wing of the d (0-15) contribution falling to the violet of the d (zero) position. In order to be generous we assume that there may be an overlap to the violet amounting to 33 per cent. of the actual area on the red side of d (zero). The corresponding f/d (10 Kv./cm.) fraction of this must then be added to the f area over the range 0-15 Kv./cm. It is easily seen that since this theoretical f/d ratio is only 0.127, it makes very little difference what assumption is made regarding the low field d intensity, as even in the most extreme case it is impossible to build up a large f area at the position f (zero-15 Kv./cm.).

It is seen that the area marked B represents the f line, the area C is equivalent to the overlap of the violet wings of the d line at fields appreciably greater than zero, plus the equivalent of area A. What remains is a large unexplained area X. No matter what assumptions are made as to the relative areas of A and D, it is impossible to increase B appreciably or in

(a) γ Peg λ 4471.

Fields in Kv./cm.	d (observed)	f (calculated)
0	66 *	0
0-15	32 †	3.0
15-25	24	7.6
25-35	18	8.2
35-50	16	8.8
50-70	12	8.8
Total absorption in } electric fields	102 $D + (C - A)$	36.4 B

d absorption $A + C + D = 168$ arbitrary units (see fig. 1).

absorption $X = 82$ units.

$$\frac{X}{A + C + D} = \frac{82}{168} = 49 \text{ per cent.}$$

* This represents $2A$.

† This includes the overlap of low field d to violet of d (zero) position, assumed to be 8 units or 33 per cent. of the portion on the red side of d (zero) position.

any other way to decrease X appreciably. In almost all the papers on this subject X is referred to as the *f* line, and indeed one of the chief arguments for Stark effect in stellar spectra has been based upon this unacceptable identification.

In a recent paper by Struve* the point which he regards as requiring clarification, and which he considers to be now "satisfactorily explained" by the hypothesis of collisional damping, is not the existence of the X hump but the fact that between the X hump and the peak of the main *d* line the profile does not more nearly approach the continuous background. *From our investigation it appears that the outstanding problem is the origin of the entire X absorption.* A suggested explanation of this absorption as due to a resonance phenomenon augmenting the *d* line will be given in Section VI.

In order to test the effect of a slight change in the position at which the continuous background is drawn in, the following alternative table for γ Peg λ 4471 is constructed. The continuous background has been lowered one arbitrary unit.

Fields in Kv./cm.	<i>d</i> (observed)	<i>f</i> (calculated)
0	66	0
0-15	31	3
15-25	21	6
25-35	15	6
35-50	11	7
50-70	6	5
Total absorption in } electric fields	84	27

d absorption $A + C + D = 150$ units.

absorption $X = 67$ units.

$$\frac{X}{A + C + D} = \frac{67}{150} = 45 \text{ per cent.}$$

3. λ 4921, 2*P*-4*D*. Regarding this anomaly of the strong absorption at and near λ 4470 in orthohelium, we would point out that there is an exact parallel in the case of parhelium. In fig. 2 are the profile of λ 4921 from one spectrogram of γ Pegasi and the corresponding chart of line positions and relative intensities. A quantitative treatment of this profile on similar lines to that just outlined results in the absorption area X remaining completely unexplained in terms of the Stark theory applied to the *D* and corresponding *F* lines, and again we are driven to the resonance argument of Section VI to account in any reasonable way for this absorption. It will be seen that the very weak *PP* line appears over a 1.5 A. range beyond its zero field (zero intensity) position at 11.3 A. to the violet of the *D* zero position.

* O. Struve, *Observatory*, 61, 53, 1938 Feb.

(b) ϵ Herc λ 4471.

Fields in Kv./cm.	d (observed)	f (calculated)
0	80 *	0
0-15	59 †	7.5
15-25	29	9.2
25-35	17	7.8
35-45	12	6.8
45-55	9	6.0
55-65	6	4.0
Total absorption in } electric fields	132 $D + (C - A)$	41.3 B

d absorption $A + C + D = 212$ units (see fig. 1).

absorption $X = 102$ units.

$$\frac{X}{A + C + D} = \frac{102}{212} = 48 \text{ per cent.}$$

(c) γ Peg λ 4921.

Fields in Kv./cm.	d (observed)	f (calculated)
0	100 ‡	0
0-20	132 §	24
20-40	97	32
Total absorption in } electric fields	229 $D + (C - A)$	56 B

D absorption $A + C + D = 329$ units.

absorption $X = 140$ units (see fig. 2).

$$\frac{X}{A + C + D} = \frac{140}{329} = 42 \text{ per cent.}$$

Some unidentified absorption lies to the violet of F (40) at approx. λ 4917.

4. λ 4026, $2p-5d$. In fig. 3 we see that a more complex system of overlapping lines is obtained, due to the f and g levels being very close together. Similar assumptions to those made in 2 with regard to the red wing of the d line lead to the calculation of the absorption to be expected

* This represents 2 A.

† This includes 15 units assumed to be the overlap of low field d to violet of d (zero) position, being 33 per cent. of the absorption (0-15) on the red of d (zero) position.

‡ 100 = 2 A.

§ This includes 33 units or 33 per cent. of the (0-20) absorption of D red wing, assumed to be the overlap on violet side of D (zero) position.

for the f and g lines at various fields. It is seen that the f line rises rapidly from zero intensity to 60 per cent. of the intensity of the d (10 Kv./cm.) line moving in towards the d (zero) position, then at successively higher fields, it moves back toward and beyond its zero position, overlapping the low field positions of the g line. Thus all the absorption due to f components will lie very compactly upon the violet wing of the d line, whereas the g absorption will be spread out weakly for more than 3 Å. to the violet beyond the f region. Quantitative work on these lines explains the smoothed profile

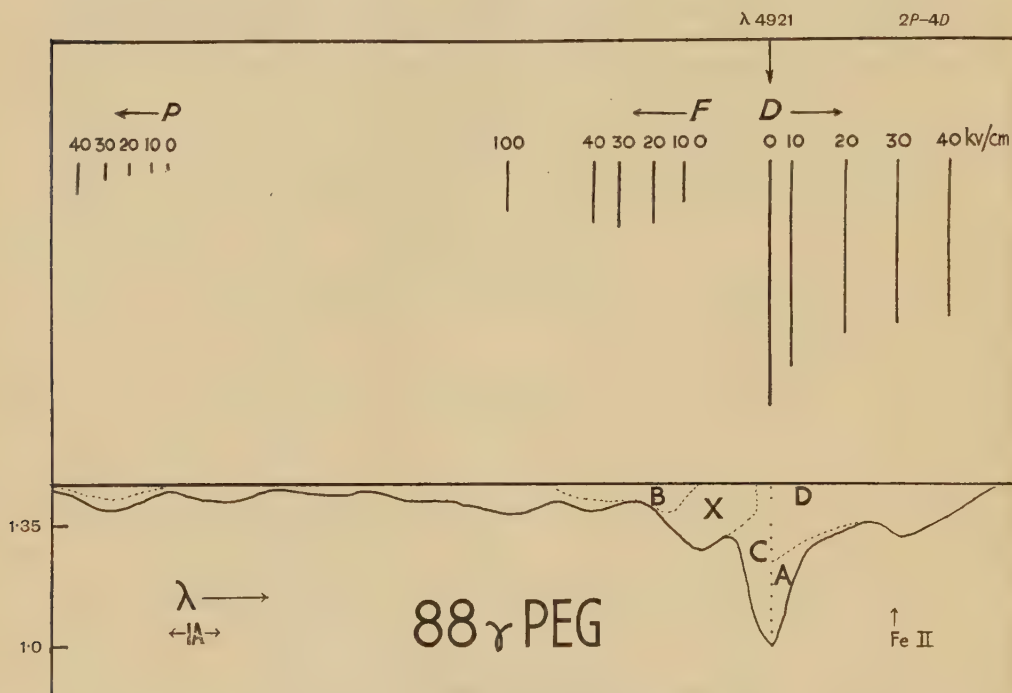


FIG. 2.

without any difficulty. There is here no remaining ambiguity or unexplained major feature and no additional hypothesis seems to be required.

Hence agreement is very satisfactory for both γ Peg and ι Herc. It may be noted that there remains unexplained a weak absorption near $\lambda 4022$ in the $g(25-30)$ region. This amounts to 12 per cent. of the total g absorption in γ Peg and 16 per cent. in ι Herc, or in terms of the total d line, 4 per cent. in γ Peg and 5 per cent. in ι Herc.

5. $\lambda 4388$, $2P-5D$. The corresponding transitions in parhelium are represented in fig. 4. The chart shows how the F lines press in very closely upon the D (zero) position for fields up to 25 Kv./cm., afterwards moving somewhat beyond the F (zero) and G (zero) positions. Thus all the F absorption to be expected corresponding to the D absorption of the wide red wing will strongly augment the violet wing of the D line and will necessarily overlap somewhat to the red of the D (zero) position, thus slightly

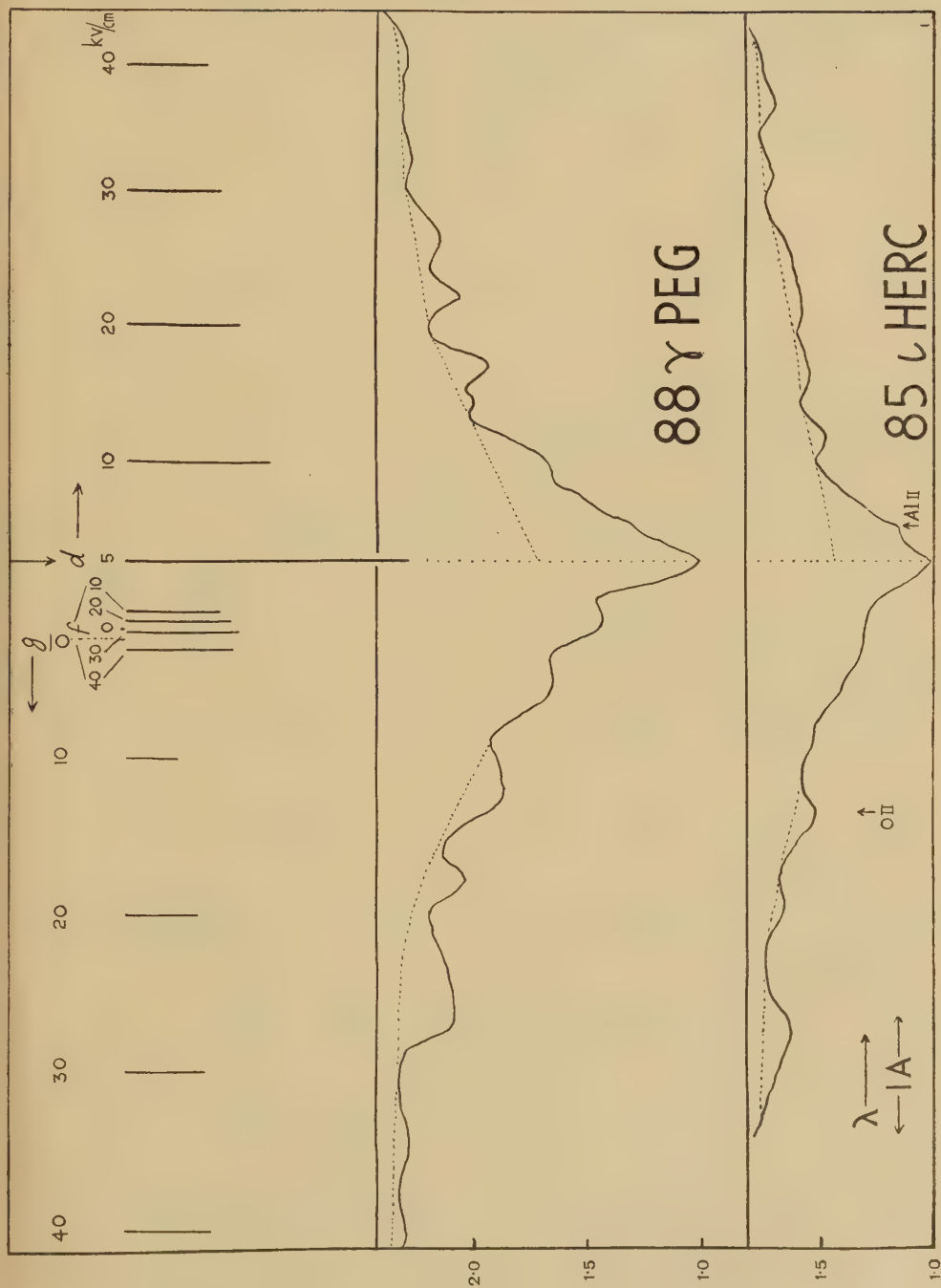


FIG. 3.

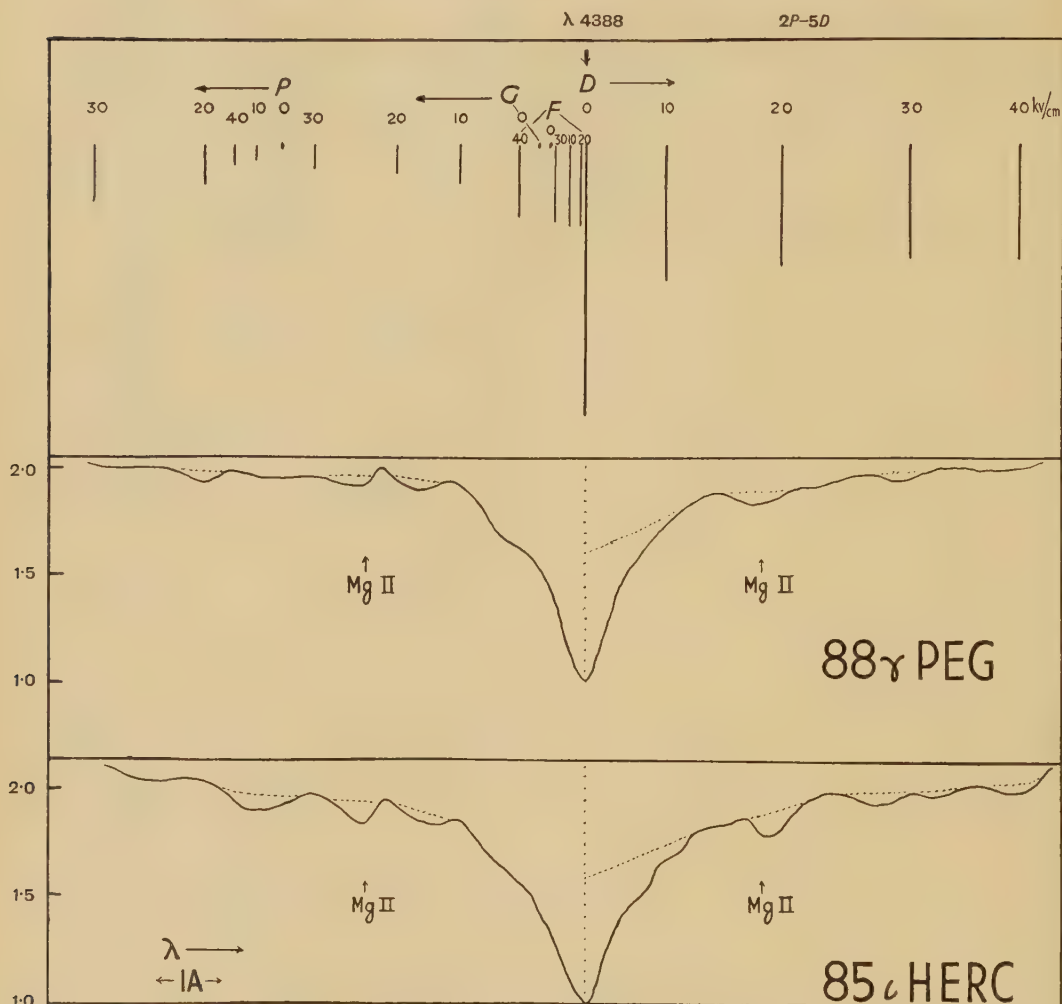


FIG. 4.

(d) γ Peg λ 4026.

Fields in Kv./cm.	d (obs.)	f (calc.)	g (calc.)
0	164	0	0
0-15	136 *	91 †	48
15-25	52	48	33
25-35	23	28	20
35-45	14	19	15
Total absorption in } electric fields	225	186	116

Total $d = 389$ arbitrary units.Total area $d + f + g(0-15) = 623$ units.Total area from actual profile to position of $g(15 \text{ Kv./cm.}) = 652$.

* This includes the assumed overlap of $d(0-15)$ to violet and excludes the assumed overlap of f to red of the d (zero) position.

† This excludes the assumed overlap of $d(0-15)$ to violet and includes the assumed overlap of f to red of the d (zero) position.

(e) ϵ Herc λ 4026.

Fields in Kv./cm.	d (obs.)	f (calc.)	g (calc.)
0	64	0	0
0-15	78 *	52 †	26
15-25	40	38	25
25-35	15	18	13
35-45	5	7	5
Total absorption in } electric fields	138	115	69

Total d = 202 arbitrary units.Total area $d + f + g(0-15)$ = 343 units.Total area from actual profile to position of $g(15 \text{ Kv./cm.})$ = 352 units.

augmenting the near red wing also. Whatever allowance we decide to make for the overlap of the F absorption to the red of D (zero) must be subtracted from the postulated value of D (low fields—15 Kv./cm.) absorption, but has no effect upon the amounts of absorptions at the fields averaging 20, 30, 40 Kv./cm. as taken directly from the red wing.

When the G and P absorption over each range of electric field is calculated in similar manner from the D absorption, it is found that the smoothed profile for 6 A. on either side of the D (zero) position is very satisfactorily explained—in other words, the violet wing can be completely reconstructed from the red wing in terms of the Stark lines associated with the F , G , P levels, no other assumptions being required.

(f) γ Peg λ 4388.

Fields in Kv./cm.	D (obs.)	F (calc.)	G (calc.)	P (calc.)
0	314	0	0	0
0-15	210	128	59	22
15-25	124	87	31	41
25-35	46	31	9	23
35-45	15	9	3	9
Total absorption in } electric fields	395	255	102	95

Total D = 709 units.Total area $D + F + G(0-15)$ = 1023 units.Total area from actual profile to position of $G(15 \text{ Kv./cm.})$ = 1057.Total area $G(15-45) + P(0-30)$ = 118.Total area from actual profile over range of $G(15)$ position to $P(30)$ position = 130.

* This includes the assumed overlap of $d(0-15)$ to violet and excludes the assumed overlap of f to red of the d (zero) position.

† This excludes the assumed overlap of $d(0-15)$ to violet and includes the assumed overlap of f to red of the d (zero) position.

(g) ϵ Herc λ 4388.

Fields in Kv./cm.	D (obs.)	F (calc.)	G (calc.)	P (calc.)
0	306	0	0	0
0-15	315	192	88	33
15-25	168	118	42	55
25-35	84	56	16	41
35-45	30	19	6	17
Total absorption in electric fields	597	385	152	146

Total $D = 903$ arbitrary units.Total $D + F + G(0-15) = 1376$.Total area from actual profile to position $G(15 \text{ Kv./cm.}) = 1433$.Total area $G(15-40) + P(0-30) = 172$.

Area available from smoothed profile over this range is approximately 250.

The exact placing of the continuous background largely determines the closeness of the fit between the actual and the theoretically expected areas of absorption.

Small residual amounts of absorption are seen to be in the overlapping region of the low field P and higher field G absorptions.

VI. CONCLUSION

The evidence presented in this paper confirms and strengthens the view that Stark effects are present in the helium spectrum of certain Class B stars. Woolley * has stated that "the position at the moment may be fairly summarized by saying that Stark effect accounts *qualitatively* for all phenomena shown by H and He lines in stellar spectra *except* the observed high central intensities of the H lines." Our evidence shows that *most* of the phenomena shown by He lines are not only qualitatively but *quantitatively* explained by Stark effect, but that *some* features, *e.g.* the heavy absorption near λ 4470 and near λ 4921, are neither qualitatively nor quantitatively so explained without introduction of features quite unrelated to the Stark effect.

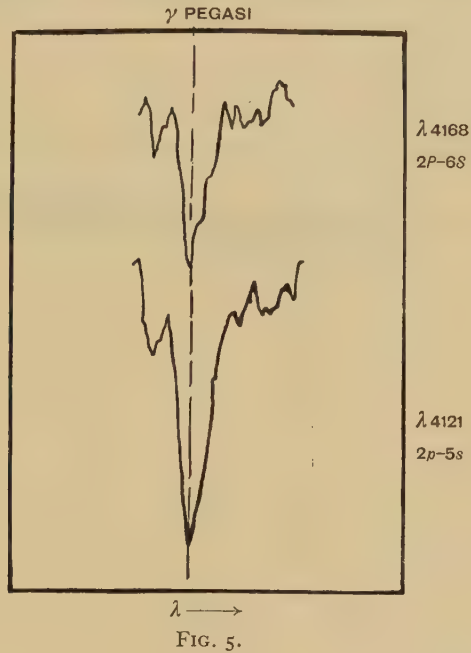
In an effort to find some reasonable explanation various possibilities have been considered. That intercombination lines between ortho- and par-helium are not the cause was seen by plotting the positions of such lines.† Nor could satisfactory explanations be based upon assumed doppler effects nor upon relative intensity changes in absorption as compared with emission.

The possibility that this large effect is due to rapid variations in the electric field in stellar atmospheres must not be overlooked. There are, however, arguments in favour of the slowly varying fields presupposed in the existing Stark effect theory. The asymmetry expected in the case of

* R. v. d. R. Woolley, *Observatory*, **60**, 239, 1937 Sept.

† Foster and Douglas, *Nature*, **134**, 417 (fig. 2), 1934 Sept. 15.

certain sharp series lines (λ_{4121} , λ_{4168}) is definitely present, and the quantitative analysis of the diffuse series lines, λ_{4026} ($2p-5d$) and λ_{4388} ($2P-5D$) together with the group of electric combination lines around each,



results in intensity profiles very nearly as expected from the theory of Stark effect in slowly varying fields.

A new line of attack must therefore be sought in explanation of the surprisingly high absorption X and the fact that it lies rather closer to the d line than would be expected if it were due to $2p-4f$. Now just these disturbing features may actually be expected—even in the absence of electric fields—if stellar conditions are such as to give (1) diffuse lines three or more angstrom units in breadth and (2) a reasonable departure from the Boltzmann distribution among atomic states. Fixing attention upon radiation characteristic of the zero field position of the electric combination line, and referring to the important cycle in fig. 6, we note that events taken in the direction indicated are certainly more probable than the reverse order.* In so far as this cycle controls the situation, there will be relatively strong net d absorption in the region of the f line, even if no electric fields are present to produce an f line, since emission accompanying the absorption will be especially weak at this point due to leakage *via* the route $4d \rightarrow 4f \rightarrow 3d$. In agreement with the observations, one should expect in varying fields the $2p-4f$ line to be augmented at zero field position

* The importance of cyclical transitions was pointed out by S. Rosseland in 1926 (*Ap. J.*, **63**, 218) in connection with the origin of emission lines in stellar spectra.

with maximum intensity shifted slightly toward the d line. There should be no sharp division between the two lines.

On the present view, we must consider that the anomalous absorption X is a secondary maximum in the d absorption. Its existence in a stellar spectrum could scarcely be considered proof that electric fields exist in the atmosphere of the star.

Nevertheless, the conditions which give broad absorption, through collisional damping, might be expected to produce local electric fields. The evidence that this is indeed the case appears to include (1) the extended

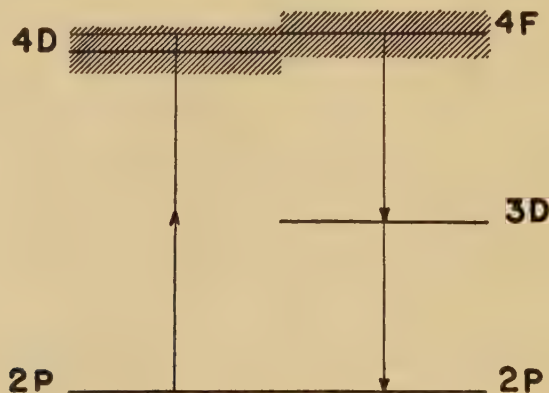


FIG. 6.

absorption on the violet side of the diffuse line. Certainly one cannot expect the d line itself to be broadened more toward the violet than toward the red, hence the more extended violet absorption actually observed cannot rest upon the above basis. It is, however, just what must be expected in the presence of electric fields, and in that case is due to well-known electric combination lines. The observed violet broadening shows on the whole a convincing excess over the red and the position of weak maxima correspond very well to the known lines, some of which are far removed from the d line, as $2p-4p$, $2p-5p$, $2p-6p$. (2) The broadened red wing of sharp series lines (fig. 5). (3) The greatly extended wings of Balmer lines explainable in no other way.

We must conclude that this entire "line" almost invariably referred to in the past as the f line is mainly due to the transition $2p-4d$ —especially near the f zero field position—but that as one proceeds towards the violet the contribution of $2p-4f$ becomes gradually the more important; and this is equally the case in interpretation of the $2P-4D$ and $2P-4F$ absorption.

It appears then that the d line is reduced to its ultimate form by emission over its entire spread, this emission being less near the f zero field position because of a relatively small net leakage from the $4d$ state to the $4f$ state and the subsequent emission of that energy in two stages. Hence the assumptions in Section V, § 2 above, by neglecting the broadening due to collisional damping, obviously overestimate the amount of absorption attributable to low electric fields, but by so doing we have thrown into

greater relief the impossibility of accounting for the X absorption on Stark theory alone.

A test of this leakage might be found by comparison of the intensity of the yellow line λ 5876 in a B giant and a B dwarf. We would expect it to be relatively somewhat weaker in the dwarf where other spectrum evidence indicates broader lines.

Similar effects must certainly be present in $2p-5d, f, g$, and $2P-5D, F, G$, but the closer packing of the levels renders impossible the separation of Stark effects and effects due mainly to this resonance phenomenon.

In view of this interpretation it would seem fortunate that the separation of $4d$ and $4f$ lines is such as to give some indication of the extent of the total absorption as it would appear if there were no associated emission.

Summary.—In the course of an investigation of some typical dwarf B stars, twelve normal helium lines have been examined, nine electric combination lines have been identified and the presence of five others found to be not incompatible with the observed profiles. From these combination lines and from the broadening to red of sharp series lines the effective electric fields are determined.

Quantitative analyses are given for the groups of lines associated with λ 4471, λ 4921, λ 4026, λ 4388. The absorption near λ 4470 is much too great and too near λ 4471 to receive the usual interpretation as $2p-4f$ at low fields.

It is suggested that this absorption (and similarly the absorption near λ 4921) is indeed part of the original broad d line overlapping the f zero field position, but while subsequent emission reduced much of the line the absorption remains relatively strong in the position of the f line owing to resonance between the $4d$ and $4f$ levels and a predominantly one-way energy cycle.

McGill University,

Montreal :

1938 December.

THE MOVING CLUSTER IN TAURUS.

W. M. Smart.

1. Thirty years ago Professor Lewis Boss completed a pioneer investigation * on the Taurus Cluster ; his results included the determination of the convergent point and of the individual parallaxes of the cluster stars. He had at his disposal the proper motions of 41 stars suspected of belonging to the cluster and the radial velocities of three of these. This research was undertaken on the eve of the publication of his *Preliminary General Catalogue*, which has been for so many years the standard source of the proper motions of the naked-eye stars.

Subsequent to Boss, several investigations of the Taurus Cluster have been made by, amongst others, Kapteyn and de Sitter,† Plummer,‡ Hertzprung § and Rasmuson ||—the last-named used the radial velocities of nine stars.

The recent publication of the *General Catalogue* of 33,342 stars by Professor Benjamin Boss, to whom and to the Carnegie Institution of Washington the whole astronomical world is under a deep debt of obligation, has provided a rich mine of accurate proper motions, and these, together with numerous determinations of radial velocities, have made it now possible to determine the spatial and kinematical characteristics of the cluster with an accuracy very much greater than was practicable in the earlier investigations referred to.

Table I contains the positional and proper motion data, taken from the *General Catalogue*, for 42 stars—substantially the same stars as used by L. Boss—to which we refer as List A. These are stars whose proper motions are mostly known now with an exceptional degree of accuracy, as the last two columns show.

Table II (List B) contains details of stars not found in L. Boss's original list and are the survivors of a somewhat larger list of stars picked out from the *General Catalogue* as possible members of the cluster. With one or two exceptions, the probable errors of the proper motions in this list are greater than for the stars in List A, but the later evidence will show that, even when the probable errors are comparatively large, all stars in the two lists may be presumed to be cluster stars. Boss's list numbered 39 presumptive members of the cluster (two were rejected from his original list of 41) ; the result of the present investigation increases the number of cluster stars to 72.

* *A.J.*, 26, 31, 1908.

† *Groningen Publ.*, 23, 1909.

‡ *M.N.*, 72, 1912.

§ *Ast. Nach.*, 209, 1918.

|| *Lund Medd.*, Ser. II, No. 26, 1921.

TABLE I

List A

No.	No. in G.C.	α		δ	$\mu_{\alpha} \cos \delta$	μ_{δ}	Probable Error	
		1950.0			(Unit : 0".001)		$\mu_{\alpha} \cos \delta$	μ_{δ}
		h	m	s				
1	4677	3	50	18	17	11	148	-28
2	4807		57	56	18	3	134	-33
3	4883	4	1	14	8	4	169	+26
4	4995		6	15	19	29	109	-32
5	5042		8	40	5	24	149	+11
6	5137		12	56	15	17	119	-29
7	5189		15	25	21	28	98	-35
8	5226		16	57	15	30	119	-24
9	5230		17	3	16	24	119	-34
10	5252		17	46	14	59	110	-23
11	5260		18	4	13	45	114	-24
12	5287		19	14	13	58	114	-25
13	5304		20	3	17	26	110	-31
14	5315		20	33	16	40	106	-29
15	5328		21	12	17	20	113	-39
16	5344		22	2	18	56	108	-45
17	5350		22	23	22	11	100	-48
18	5351		22	26	22	5	113	-54
19	5354		22	36	17	49	111	-29
20	5356		22	46	15	50	110	-29
21	5370		23	19	22	42	108	-47
22	5375		23	30	15	30	114	-23
23	5412		25	2	21	31	99	-38
24	5425		25	33	14	38	110	-23
25	5430		25	42	19	4	112	-38
26	5433		25	43	15	51	105	-28
27	5436		25	48	15	46	105	-26
28	5443		26	2	12	56	111	-13
29	5467		27	17	15	32	104	-25
30	5480		27	42	16	5	110	-28
31	5482		27	48	15	35	105	-24
32	5483		27	48	13	37	106	-22
33	5517		29	0	15	45	104	-28
34	5558		31	0	14	44	102	-28
35	5643		35	18	15	56	95	-24
36	5645		35	22	12	25	101	-12
37	5665		36	24	7	46	89	+2
38	5666		36	25	15	49	84	-18
39	5767		41	39	11	3	100	-14
40	5907		48	27	18	45	81	-35
41	6085	4	56	52	15	51	91	-24
42	6300	5	6	34	9	46	65	-4

2. Boss's determination of the convergent point was partly empirical and partly statistical. The method, originally due to Charlier, adopted in the present investigation is as follows:—

TABLE II

List B

No.	No. in G.C.	α		δ	$\mu_\alpha \cos \delta$		μ_δ	Probable Error	
		1950.0			(Unit : 0".001)			(Unit : 0".0001)	
		h	m	s	o	'			
43	4790	3	56	56	10	11	171	+ 8	148
44	4796		57	44	20	14	145	- 51	92
45	4964	4	4	52	15	2	135	- 24	70
46	5023		7	49	18	18	112	- 45	74
47	5106		11	32	22	20	93	- 27	94
48	5129		12	46	20	42	104	- 50	91
49	5234		17	9	13	55	116	- 20	31
50	5246		17	30	18	37	111	- 44	70
51	5271		18	34	20	55	109	- 28	78
52	5273		18	38	18	18	103	- 27	87
53	5277		18	45	14	18	116	- 28	109
54	5300		19	52	16	40	145	- 48	86
55	5313		20	29	24	17	93	- 43	49
56	5330		21	16	21	37	109	- 57	76
57	5339		21	36	16	46	116	- 31	90
58	5367		23	15	15	25	113	- 37	103
59	5372		23	20	21	22	101	- 35	69
60	5445		26	8	16	3	99	- 41	71
61	5455		26	37	17	45	120	- 35	124
62	5470		27	22	19	44	93	- 39	68
63	5487		27	55	16	2	101	- 44	55
64	5531		29	25	5	18	114	+ 12	75
65	5556		30	58	13	9	117	- 16	76
66	5616		33	49	15	46	101	- 30	103
67	5802		43	15	11	37	72	- 5	55
68	5885		47	30	17	7	96	- 42	82
69	5906		48	27	10	59	108	+ 7	116
70	5923		49	0	13	34	112	- 32	156
71	5984		52	2	19	24	64	- 39	91
72	6158	5	0	6	21	31	67	- 43	13

Let (X, Y, Z) denote the equatorial linear components, in km./sec., of the common cluster motion, V . Then for each star we have the pair of equations :

$$-X \sin \alpha + Y \cos \alpha = \frac{\kappa}{p} \mu_\alpha \cos \delta, \quad (1)$$

$$-X \cos \alpha \sin \delta - Y \sin \alpha \sin \delta + Z \cos \delta = \frac{\kappa}{p} \mu_\delta, \quad (2)$$

in which $\kappa = 4.738$ and p is the star's parallax.

Let $\xi = \mu_\alpha \cos \delta$, $\eta = \mu_\delta$,

and, eliminating p from (1) and (2), we have

$$aX + bY - cZ = 0, \quad (3)$$

where

$$a = \xi \cos \alpha \sin \delta - \eta \sin \alpha, \quad (4)$$

$$b = \xi \sin \alpha \sin \delta + \eta \cos \alpha, \quad (5)$$

$$c = \xi \cos \delta. \quad (6)$$

We rewrite (3) in the form

$$a\left(\frac{X}{Y}\right) - c\left(\frac{Z}{Y}\right) + b = 0. \tag{7}$$

A least-squares solution of (7) gives the values of X/Y and Z/Y .
If (A, D) denote the co-ordinates of the convergent point,

$$X = V \cos A \cos D, \quad Y = V \sin A \cos D, \quad Z = V \sin D, \tag{8}$$

from which

$$\tan A = Y/X, \tag{9}$$

$$\tan D = Z/(X^2 + Y^2)^{\frac{1}{2}}. \tag{10}$$

TABLE III

No.	<i>a</i>	<i>b</i>	<i>c</i>	Weight	No.	<i>a</i>	<i>b</i>	<i>c</i>	Weight
1	47.0	21.9	141.3	2	37	2.4	11.9	88.2	1
2	49.4	19.0	127.4	2	38	24.9	15.0	80.8	2
3	-10.9	33.5	167.3	1	39	19.6	13.4	98.1	2
4	45.4	16.8	102.8	3	40	41.3	14.1	76.7	3
5	-3.1	17.5	148.4	2	41	29.8	17.4	87.5	2
6	40.1	15.0	114.8	2	42	6.4	9.8	64.0	2
7	47.2	16.8	91.2	2	43	8.5	30.1	168.3	$\frac{1}{4}$
8	35.4	18.3	114.7	3	44	69.4	17.2	136.0	$\frac{1}{2}$
9	45.1	15.4	114.1	1	45	37.9	19.0	130.4	$\frac{3}{4}$
10	33.0	15.8	106.3	2	46	56.3	9.8	106.3	$\frac{3}{4}$
11	33.3	14.2	110.7	2	47	40.1	19.1	86.0	$\frac{1}{2}$
12	34.3	14.2	110.6	2	48	62.7	13.2	97.2	$\frac{1}{2}$
13	42.0	16.7	104.9	3	49	30.1	16.4	112.6	2
14	39.1	15.4	101.5	2	50	55.0	13.0	105.2	$\frac{3}{4}$
15	49.5	14.3	107.9	3	51	42.0	23.2	101.8	$\frac{3}{4}$
16	55.5	13.2	102.2	2	52	38.2	17.7	97.7	$\frac{1}{2}$
17	59.3	14.6	92.6	2	53	37.6	13.9	112.4	$\frac{1}{2}$
18	66.7	16.5	104.8	2	54	61.0	17.4	138.9	$\frac{1}{2}$
19	40.4	19.1	105.7	3	55	55.1	16.6	84.8	1
20	38.7	15.5	105.8	2	56	68.5	12.7	101.4	$\frac{3}{4}$
21	59.9	18.8	99.7	3	57	42.1	17.5	111.1	$\frac{1}{2}$
22	33.4	18.4	109.9	2	58	46.0	12.4	108.9	$\frac{1}{4}$
23	49.5	18.0	92.1	1	59	46.9	19.2	94.0	$\frac{3}{4}$
24	32.2	16.3	106.4	2	60	48.5	8.7	95.1	$\frac{3}{4}$
25	49.5	18.4	105.8	3	61	46.6	19.7	114.2	$\frac{1}{4}$
26	37.1	14.9	101.0	3	62	48.2	13.6	87.5	$\frac{3}{4}$
27	35.2	15.7	101.0	3	63	51.4	8.5	97.1	1
28	21.8	17.6	108.2	2	64	-7.1	14.3	113.5	$\frac{3}{4}$
29	34.0	15.7	100.2	2	65	24.9	18.6	114.0	$\frac{3}{4}$
30	37.8	17.1	105.7	2	66	38.0	14.6	97.2	$\frac{1}{4}$
31	33.1	16.5	101.1	2	67	9.5	12.1	70.6	1
32	30.0	14.3	103.0	2	68	48.6	13.7	91.8	$\frac{3}{4}$
33	36.7	15.2	100.0	2	69	-0.3	21.8	106.1	$\frac{1}{4}$
34	35.7	13.4	98.6	3	70	38.5	15.2	108.9	$\frac{1}{4}$
35	31.8	15.6	91.4	2	71	43.5	8.9	60.4	$\frac{1}{4}$
36	19.1	15.9	98.7	2	72	47.9	12.7	62.3	2

Table III contains the values * of a , b and c for all the stars in Lists A and B. To each star a weight has been assigned, based roughly on the probable errors given in Tables I and II, for use in the equation of condition (7).

The procedure adopted was to make a solution of (7) from the stars in List A and to reject stars from the supplementary list with large residuals ; in this way List B was obtained. A final solution was made using the stars in Lists A and B.

The results of the solution for List A are as follows :—

$$\begin{aligned} X/Y &= -0.02669 \pm 0.01350, \\ Z/Y &= 0.14633 \pm 0.00510, \end{aligned}$$

from which, by (9) and (10),

$$\begin{aligned} A &= 91^\circ 32' \pm 0^\circ.77, \\ D &= 8^\circ 19' \pm 0^\circ.29. \end{aligned}$$

The results of the solution for stars in the combined lists, derived from the normal equations (weight 111)

$$\begin{aligned} 183,135.7(X/Y) - 426,651.2(Z/Y) + 66,980.2 &= 0, \\ 426,651.2(X/Y) - 1,182,151.2(Z/Y) + 182,988.5 &= 0, \end{aligned}$$

are as follows :—

$$\begin{aligned} X/Y &= -0.03216 \pm 0.01160, \\ Z/Y &= 0.14318 \pm 0.00456, \end{aligned}$$

from which

$$\begin{aligned} A &= 91^\circ 50' \pm 0^\circ.66, \\ D &= 8^\circ 9' \pm 0^\circ.26. \end{aligned}$$

These last values are taken as the co-ordinates of the convergent point for the subsequent calculations. It is to be noted that these co-ordinates are referred to the mean equinox for 1950.0.

Boss's results, brought up to 1950.0, were

$$A = 92^\circ 49', \quad D = 6^\circ 55',$$

with estimated probable errors of $\pm 1^\circ.5$ in A and $\pm 0^\circ.3$ in D ; Rasmuson's results were

$$A = 94^\circ.0 \pm 1^\circ.1, \quad D = 7^\circ.0 \pm 1^\circ.1.$$

Table IV contains the values of the position angle, θ , of the total proper motion, μ , as observed and as calculated with the adopted co-ordinates of the convergent point. The weights used in the solution of (7) are reproduced for convenience. For the stars in List A (Nos. 1-42) the weighted average residual (taken without regard to sign) is $1^\circ.78$; for the stars in List B it is $3^\circ.66$; and for all the 72 stars it is $2^\circ.12$. A scrutiny of Table IV will show that, with all due consideration of the probable errors of the proper motions, no star in the Lists A and B should be rejected off-hand as not being a probable member of the cluster.

* The unit is $0''.001$.

TABLE IV
Position-angles of the Total Proper Motions

No.	θ		O - C	Weight		No.	θ		O - C	Weight
	Obs.	Calc.					Obs.	Calc.		
1	100.7	100.7	0.0	2		37	88.7	87.7	+ 1.0	1
2	103.8	103.3	+0.5	2		38	102.1	106.4	- 4.3	2
3	81.2	87.5	-6.3	1		39	96.5	95.0	+ 1.5	2
4	106.4	106.9	-0.5	3		40	113.4	116.2	- 2.8	3
5	94.2	97.0	-2.8	2		41	104.8	112.0	- 7.2	2
6	103.7	101.0	+2.6	2		42	93.5	95.5	- 2.0	2
7	109.6	111.9	-2.3	2		43	87.3	90.7	- 3.4	$\frac{1}{4}$
8	101.4	102.0	-0.6	3		44	109.4	106.4	+ 3.0	$\frac{1}{2}$
9	106.0	103.7	+2.3	1		45	100.1	97.0	+ 3.1	$\frac{3}{4}$
10	101.8	101.2	+0.6	2		46	111.9	105.3	+ 6.3	$\frac{3}{4}$
11	101.8	98.9	+2.9	2		47	106.2	112.3	- 6.1	$\frac{1}{2}$
12	102.3	99.7	+2.6	2		48	115.7	112.4	+ 3.3	$\frac{1}{2}$
13	105.8	106.1	-0.3	3		49	99.8	99.0	+ 0.8	2
14	105.3	104.9	+0.4	2		50	111.6	107.7	+ 3.9	$\frac{3}{4}$
15	109.0	106.3	+2.7	3		51	104.4	111.8	- 7.4	$\frac{3}{4}$
16	112.6	109.2	+3.4	2		52	104.7	107.4	- 2.7	$\frac{1}{2}$
17	115.6	115.0	+0.6	2		53	103.6	100.0	+ 3.6	$\frac{1}{2}$
18	115.6	114.9	+0.7	2		54	108.3	104.7	+ 3.6	$\frac{1}{2}$
19	104.6	107.4	-2.8	3		55	114.8	117.8	- 3.0	1
20	104.8	103.6	+1.2	2		56	117.6	113.8	+ 3.8	$\frac{3}{4}$
21	113.5	116.1	-2.6	3		57	105.0	105.3	- 0.3	$\frac{1}{2}$
22	101.4	103.2	-1.8	2		58	108.1	102.7	+ 5.4	$\frac{1}{4}$
23	111.0	114.6	-3.6	1		59	109.1	113.9	- 4.8	$\frac{3}{4}$
24	101.8	101.9	-0.1	2		60	112.5	104.8	+ 7.7	$\frac{3}{4}$
25	108.7	110.3	-1.6	3		61	106.3	108.2	- 1.9	$\frac{1}{4}$
26	105.0	104.1	+0.9	3		62	112.7	112.2	+ 0.5	$\frac{3}{4}$
27	103.9	104.1	-0.2	3		63	113.6	104.9	+ 8.7	1
28	96.7	98.2	-1.5	2		64	84.0	81.9	+ 2.1	$\frac{3}{4}$
29	103.2	103.9	-0.7	2		65	97.8	97.8	0.0	$\frac{3}{4}$
30	104.3	105.2	-0.9	2		66	106.5	105.7	+ 0.8	$\frac{1}{4}$
31	102.9	104.3	-1.4	2		67	94.0	97.6	- 3.6	1
32	101.8	100.1	+1.7	2		68	113.7	111.5	+ 2.2	$\frac{3}{4}$
33	105.1	104.7	+0.4	2		69	86.3	96.5	- 10.2	$\frac{1}{4}$
34	105.3	103.1	+2.2	3		70	106.0	103.0	+ 3.0	$\frac{1}{4}$
35	104.2	106.4	-2.2	2		71	121.4	119.0	+ 2.4	$\frac{1}{4}$
36	96.8	98.4	-1.6	2		72	122.7	127.0	- 4.3	2

3. We consider now the observed radial velocities, ρ .

We have, for each star,

$$X \cos \alpha \cos \delta + Y \sin \alpha \cos \delta + Z \sin \delta + K = \rho, \quad (11)$$

where K is the K term. Theoretically it is possible from this equation to determine the values of X , Y , Z and K , but, owing to the comparatively close grouping of the stars and the large uncertainties in the values of ρ , it is impracticable to make a solution of (11) in the ordinary way.

We adopt the position of the convergent point as determined from the proper motions and (11) then becomes

$$V \cos \lambda + K = \rho, \quad (12)$$

where λ is a star's angular distance from the convergent point. For the reasons given in connection with (11), the formula (12) leads to no determinate solution and we are forced to adopt a value of K . In a recent paper * Mr. Green and I obtained the values $+0.03$, -0.59 , -0.97 and -0.19 of the K term for A, F, G and K types (to which the cluster stars belong) from a study of the solar motion based on radial velocities. It is most likely that these values have little actual significance and the best procedure

TABLE V
Radial Velocities

No.	H.D.	ρ	P.E.	Weight	λ	$\rho \sec \lambda$	Residual
1	24357	31.6	± 0.6	2	34 34	38.4	-4.1
3	25570	34.8	± 0.6	2	31 13	40.7	-1.8
5	26462	35.8	± 2.6	$\frac{1}{2}$	29 35	41.2	-1.1
8	27371	38.5	± 0.3	3	27 58	43.6	+1.1
9	27383	36.6	± 0.9	2	28 8	41.5	-1.0
13	27697	38.3	± 0.3	3	27 43	43.3	+0.8
14	27749	36.4	...	$\frac{1}{2}$	27 24	41.0	-1.5
15	27819	36.6	± 1.9	1	27 26	41.2	-1.3
16	27901	36.1	± 0.4	3	27 43	40.8	-1.7
17	27934	41.0	± 1.2	1	28 52	46.8	+4.3
19	27962	35.2	± 0.2	3	27 15	39.6	-2.9
20	27991	38.5	± 0.9	2	26 40	43.1	+0.6
23	28226	35.9	± 1.4	1	28 3	40.7	-1.8
25	28305	38.8	± 0.3	3	26 57	43.5	+1.0
26	28307	40.0	± 0.6	2	25 58	44.4	+1.9
27	28319	42.6	...	$\frac{1}{2}$	25 57	47.3	+4.8
28	28355	33.0	± 1.7	1	25 20	36.5	-6.0
30	28527	37.7	± 2.2	$\frac{1}{2}$	25 36	41.8	-0.7
31	28546	39.2	± 0.6	2	25 27	43.4	+0.9
32	28556	39.1	± 0.9	2	25 2	43.2	+0.7
35	29375	38.2	± 1.4	1	23 48	41.7	-0.8
39	30034	39.3	± 0.7	2	21 17	42.2	-0.3
41	31845	43.9	± 1.0	2	18 51	46.4	+3.9
42	33254	38.8	± 1.0	2	15 7	40.2	-1.9
46	26345	31.0	...	$\frac{1}{2}$	30 45	36.8	-5.7
55	27731	34.0	...	$\frac{1}{2}$	30 12	39.4	-3.1
57	27859	43.8	± 0.7	2	27 9	49.2	+6.7
58	28034	40.9	± 2.0	1	26 26	45.7	+3.2
61	28406	34.0	...	$\frac{1}{2}$	26 19	38.0	-4.5
63	28568	43.1	± 0.6	2	25 30	47.7	+5.5
67	30210	38.8	± 1.5	1	21 0	41.5	-1.0
68	30676	41	...	$\frac{1}{2}$	21 24	44.0	+1.5
69	30810	38.6	± 1.3	1	19 38	41.0	-1.5
70	30869	38.3	± 1.6	1	19 57	40.8	-1.7
72	32301	43.6	± 1.4	1	21 0	46.7	+4.2

* M.N., 96, 476, 1936.

seems to be to neglect K in (12). The equation of condition for the radial velocities then becomes

$$V = \rho \sec \lambda. \quad (13)$$

Of the 72 stars in Lists A and B, 35 appear in *Moore's Catalogue of Radial Velocities*.^{*} The details, including the catalogue probable errors, are shown in Table V. The radial velocities of several stars in the cluster are variable and several have no probable errors assigned. As in the case of the proper motions a rough weighting, based on Moore's probable errors, has been adopted. The solution of (13) is given by

$$V = 42.5 \pm 0.30.$$

The residuals are shown in the table.

The corresponding value of V obtained by L. Boss in 1908 from three radial velocities was 45.6.

The linear equatorial components, in km./sec., of the solar motion, of 19.5 km./sec. and apex (270° , $+30^\circ$), in the direction of the convergent point are respectively 0, -16.9 and 9.7 ; the linear components of the cluster motion relative to the Sun are -1.4 , 42.0 and 5.9 . Hence, relative to the mean of the stars in the neighbourhood of the Sun, the components for the cluster are -1.4 , 58.9 and -3.8 respectively, which give the linear velocity of the cluster as 59.0 km./sec. in the direction R.A. 90° , Dec. $-3^\circ.7$, or, in galactic co-ordinates, longitude 172° , latitude -7° .

4. With V determined we can now calculate the parallax, p , of each star from the formula

$$p = \frac{\kappa\mu}{V \sin \lambda}, \quad (14)$$

where $\kappa = 4.738$ and μ is the total proper motion.

Tables VI and VII give the results for stars in Lists A and B separately; for stars of adopted weights 3, 2 and 1 it seems justifiable to calculate the parallaxes to a fourth place of decimals. The spectral types (taken from the *General Catalogue*) and the absolute magnitudes, M , as calculated from the parallaxes and the apparent magnitudes, m , in the usual way are given in the last two columns of these tables.

The figure on p. 178—the Hertzsprung-Russell diagram—shows the distribution of absolute magnitude according to spectral type. For a given type there is a considerable spread † in the absolute magnitudes, which is clearly real.

We have, using (14),

$$M = m + 5 + 5c \log_e \left(\frac{\kappa\mu}{V \sin \lambda} \right), \quad (15)$$

where $c = 0.4343$. An error in V merely shifts the zero of absolute magnitudes without any alteration in the spread. An error in the position

^{*} *Lick Publ.*, 18, 1932.

† An exception concerns the group of four K-type giants whose absolute magnitudes are almost identical.

of the convergent point will lead to a different series of values for λ for the several stars and consequently to a different spread of the absolute

TABLE VI
Parallaxes and Absolute Magnitudes, List A

No.	μ (Unit: 0".001)	λ	Parallax	Weight	m	Spectral Type	Absolute Magnitude, M
		$^{\circ}$ $'$	$''$				
1	150.6	34 34	0.0296	2	5.96	Fo	3.32
2	138.0	32 58	283	2	5.76	Fo	3.02
3	171.0	31 13	368	1	5.48	Fo	3.31
4	113.6	31 26	243	3	5.67	G5	2.60
5	149.4	29 35	338	2	5.71	Fo	3.35
6	122.5	28 52	283	2	6.35	F5	3.61
7	104.0	30 3	232	2	5.56	A5	2.38
8	121.4	27 58	289	3	3.86	Ko	1.16
9	123.8	28 8	293	1	6.86	Go	4.19
10	112.4	27 40	270	2	5.27	Fo	2.43
11	116.5	27 23	282	2	6.14	F2	3.39
12	116.7	27 9	285	2	5.76	A3	3.04
13	114.3	27 43	274	3	3.93	Ko	1.12
14	109.9	27 24	266	2	5.68	A2	2.81
15	119.5	27 26	289	3	4.84	A5	2.15
16	117.0	27 43	280	2	5.96	Fo	3.20
17	110.9	28 52	256	2	4.36	A3	1.40
18	125.2	28 50	290	2	5.42	Fo	2.73
19	114.7	27 15	279	3	4.24	A2	1.47
20	113.8	26 40	283	2	6.39	F8	3.65
21	117.8	28 53	272	3	4.40	A5	1.57
22	116.3	26 26	291	2	4.60	A5	1.92
23	106.0	28 3	251	1	5.74	A5	2.74
24	112.4	25 46	288	2	5.97	Fo	3.27
25	118.3	26 57	291	3	3.63	Ko	0.95
26	108.7	25 58	277	3	4.04	Ko	1.25
27	108.2	25 57	276	3	3.62	Fo	0.82
28	111.8	25 20	291	2	5.12	A5	2.44
29	107.0	25 33	277	2	5.70	Fo	2.91
30	113.5	25 36	293	2	4.84	A5	2.17
31	107.7	25 27	279	2	5.49	A5	2.72
32	108.3	25 2	285	2	5.49	Fo	2.77
33	107.7	25 12	282	2	6.04	Fo	3.29
34	105.8	24 30	284	3	4.75	A5	2.02
35	98.0	23 48	271	2	5.80	Fo	2.96
36	101.7	23 1	290	2	4.30	A3	1.61
37	89.0	22 31	259	1	5.55	Fo	2.62
38	85.9	23 30	240	2	4.85	A3	1.75
39	101.0	21 17	310	2	5.35	A3	2.81
40	88.2	21 51	264	3	5.12	Fo	2.23
41	94.1	18 51	325	2	6.70	Fo	4.26
42	65.0	15 7	277	2	5.42	A2	2.63

magnitudes of a given type. But it is inconceivable that the position of the convergent point is in error by such an amount that would produce a

substantial change in the observed spread. If we assume that the visual magnitudes and spectral types are determined with reasonable accuracy, there remains the effect of the probable errors in μ . Due to this cause alone we have from (15)

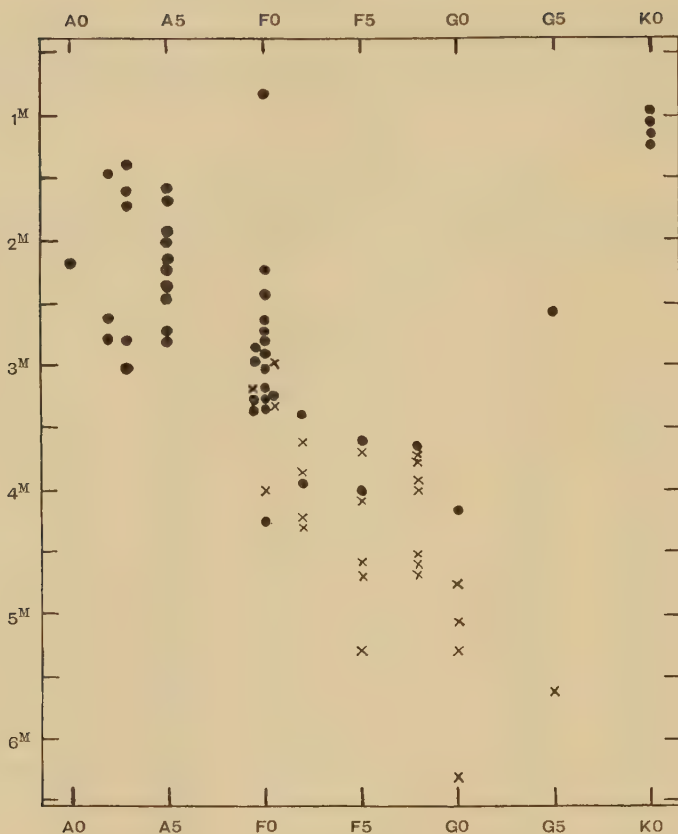
$$\Delta M = 5c\Delta\mu/\mu. \quad (16)$$

For stars in List A the average probable error in μ may be put as high as $0''.004$; taking $0''.1$ to be a typically low value of μ (Table VI), we find

TABLE VII
Parallaxes and Absolute Magnitudes, List B

No.	μ (Unit: $0''.001$)	λ	Parallax	Weight	m	Spectral Type	Absolute Magnitude, M
43	171.2	32 14	0.036	$\frac{1}{4}$	6.42	F2	4.2
44	153.7	33 34	31	$\frac{1}{2}$	8.8	G0	6.3
45	137.1	30 34	30	$\frac{3}{4}$	5.94	F0	3.3
46	120.7	30 45	26	$\frac{3}{4}$	6.62	F8	3.7
47	96.8	31 13	21	$\frac{1}{2}$	7.00	F2	3.6
48	115.4	30 53	25	$\frac{1}{2}$	8.1	G0	5.1
49	117.7	27 37	283	2	5.59	F0	2.85
50	119.4	28 37	28	$\frac{3}{4}$	5.96	F0	3.2
51	112.5	29 11	26	$\frac{3}{4}$	6.88	F8	3.9
52	106.5	28 16	25	$\frac{1}{2}$	6.74	F5	3.7
53	119.3	27 18	29	$\frac{1}{2}$	6.71	F0	4.0
54	152.7	27 33	37	$\frac{1}{2}$	7.8	G5	5.6
55	102.5	30 12	227	1	7.23	F5	4.01
56	123.0	28 53	28	$\frac{3}{4}$	8.0	F5	5.3
57	120.1	27 9	29	$\frac{1}{2}$	8.0	G0	5.3
58	118.9	26 26	30	$\frac{1}{4}$	7.43	G0	4.8
59	106.9	28 20	25	$\frac{3}{4}$	7.51	F8	4.5
60	107.2	25 57	27	$\frac{3}{4}$	6.58	F8	3.8
61	125.0	26 19	31	$\frac{1}{4}$	7.06	F8	4.5
62	100.8	26 51	25	$\frac{3}{4}$	7.15	F5	4.1
63	110.2	25 30	285	1	6.66	F2	3.94
64	114.6	24 28	31	$\frac{3}{4}$	6.43	F2	3.9
65	118.1	24 10	32	$\frac{3}{4}$	6.66	F2	4.2
66	104.9	24 5	29	$\frac{1}{4}$	6.67	F8	4.0
67	72.2	21 0	225	1	5.43	A0	2.19
68	104.8	21 24	32	$\frac{3}{4}$	7.22	F8	4.7
69	108.2	19 38	36	$\frac{1}{4}$	6.96	F5	4.7
70	116.5	19 57	38	$\frac{1}{4}$	6.70	F5	4.6
71	75.0	21 26	23	$\frac{1}{4}$	6.24	F0	3.0
72	79.6	21 0	248	2	4.70	A5	1.67

from (16) that the probable error of M is not likely to exceed $5c/\mu$ times the probable error of μ —that is, about $\pm 0^m.09$. The observed spread for a given spectral type for the stars with well-determined parallaxes in List A is clearly not due in any substantial measure to the effects of this small probable error and we conclude that the dispersion in absolute magnitude is real.



Hertzsprung-Russell Diagram.

Stars of weights 1 to 3 are represented by full circles, all other stars by crosses.

5. We proceed now to determine the distribution of the stars in the cluster. Table VIII gives the heliocentric equatorial co-ordinates of each star in parsecs, referred to the mean equinox of 1950.0. A preliminary determination was made using only the stars of weights 2 and 3. The mean heliocentric co-ordinates were found to be (13.7, 31.4, 10.0), and the distances of each star from this point was calculated. It was then evident that the cluster was strongly concentrated about this point, the distances of 25 stars out of the 40 originally selected by weight being less than 3 parsecs. Taking only these 25 stars, we find that the mean co-ordinates are now (13.8, 31.2, 9.6), and this point is taken to be the centre of the cluster. The parallax of the centre is $0''.0282$. The distance, R , of each star from the centre is given in columns 5 and 10 of Table VIII.

Table IX shows the number of stars between the limits (in parsecs) 0 to 1, 1 to 2, . . . from the centre and the corresponding mean density (number of stars per cubic parsec) for stars (i) of weights 1 to 3 and (ii) of all weights; a star at distance 2.0, for example, is counted as a half in the range 1 to 2 and in the range 2 to 3.

TABLE VIII
Heliocentric Co-ordinates of the Cluster Stars

No.	<i>x</i>	<i>y</i>	<i>z</i>	<i>R</i>	No.	<i>x</i>	<i>y</i>	<i>z</i>	<i>R</i>
1	17.3	27.2	9.9	5.3	37	13.7	35.7	5.2	6.3
2	17.1	29.0	11.0	4.2	38	14.3	37.4	11.4	6.5
3	13.3	23.4	3.8	9.7	39	10.6	29.8	6.2	4.9
4	18.5	34.2	13.8	7.0	40	11.0	34.2	12.2	4.8
5	13.8	26.0	2.8	8.6	41	8.1	28.5	8.4	6.4
6	15.4	30.5	9.3	1.8	42	8.2	34.5	6.1	7.3
7	17.8	36.1	15.8	8.9	43	14.1	23.7	5.0	8.8
8	14.5	30.2	9.3	1.3	44	15.4	26.1	11.2	5.6
9	14.2	29.5	9.6	1.7	45	15.5	28.2	8.6	3.6
10	15.5	32.3	9.6	2.0	46	16.9	31.8	11.9	3.9
11	14.8	31.0	8.4	1.6	47	20.3	39.5	18.2	13.6
12	14.5	30.8	8.5	1.4	48	16.8	33.3	15.4	6.9
13	14.7	31.5	10.9	1.6	49	14.9	31.0	8.5	1.6
14	15.2	32.7	10.8	2.4	50	14.8	30.8	11.5	2.2
15	13.8	29.9	10.3	1.5	51	15.6	32.8	13.9	4.9
16	14.0	30.7	11.5	2.0	52	16.2	34.2	12.5	4.8
17	14.9	33.0	14.8	5.6	53	14.3	30.2	8.5	1.6
18	13.2	29.1	13.0	4.0	54	11.0	23.5	7.8	8.4
19	14.0	31.0	11.0	1.4	55	16.9	36.4	18.1	10.4
20	14.0	31.0	9.7	0.3	56	13.7	29.8	13.0	3.7
21	13.9	31.0	14.2	4.6	57	13.6	29.7	9.8	1.5
22	13.6	30.2	9.2	1.1	58	13.2	29.5	8.9	1.9
23	14.9	33.9	14.6	5.8	59	15.2	33.8	14.5	5.7
24	13.5	30.7	8.8	1.0	60	14.0	32.2	10.1	1.1
25	13.0	29.7	11.2	2.3	61	12.0	27.8	9.7	3.8
26	13.9	31.9	9.9	0.7	62	14.9	34.8	13.6	5.5
27	14.0	32.0	9.9	0.8	63	13.2	31.0	9.7	0.6
28	13.4	30.8	7.7	2.0	64	12.4	29.8	3.0	6.9
29	13.7	32.0	9.7	0.8	65	11.4	28.0	7.1	4.7
30	12.9	30.2	9.5	1.4	66	12.3	31.2	9.5	1.5
31	13.5	31.7	9.6	0.6	67	14.3	41.2	8.9	10.3
32	13.4	31.4	8.2	1.5	68	9.3	28.4	9.2	5.3
33	13.2	31.5	9.6	0.6	69	8.4	26.0	5.3	8.6
34	12.9	31.5	8.9	1.2	70	7.8	24.2	6.1	9.9
35	12.8	33.2	10.2	2.1	71	12.0	39.4	14.5	9.7
36	12.2	31.4	7.4	2.7	72	9.7	36.3	14.8	8.4

TABLE IX

Limits (parsecs)	Weights 1-3		All Weights	
	No. of Stars	Density	No. of Stars	Density
0-1	6½	1.55	7½	1.79
1-2	15	1.19	20	1.59
2-3	5½	0.27	6½	0.31
3-4	½	0.02	4½	0.16
4-5	4½	0.12	7½	0.20
5-6	3	0.06	7	0.15
6-7	3½	0.06	5½	0.10
7-8	1½	0.02	1½	0.02
> 8	4	...	12	...

Beyond a distance of two parsecs from the centre the density falls off rapidly.

I am indebted to Mr. T. R. Tannahill for valuable assistance in checking the various computations.

6. *Summary*.—The accurate proper motions of the new *General Catalogue* have been utilised in determining the convergent point of the Taurus Cluster ; it is found to be R.A. $91^{\circ} 50'$, Dec. $8^{\circ} 9'$. Using proper motion as a criterion, the number of stars belonging to the cluster is 72. The speed of the cluster has been found to be 42.5 km./sec. from the radial velocities of 35 members of the cluster. The parallaxes and absolute magnitudes of the individual stars are given in Tables VI and VII. The cluster is found to be strongly condensed towards its centre, the number of stars within a sphere of radius 3 parsecs being 34 out of a total membership of 72.

University Observatory,
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ON THE ROTATIONS OF THE PLANETS.

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Introduction.—In a recent paper on the origin of satellites * it has been shown by the writer that the rotational instability of the primitive planets must be invoked to account for the existence of bodies as small as the satellites and the terrestrial planets. Such a mode of generation of the inner planets is also required to explain their differences of composition, a feature of the solar system of which a theory must give some explanation if it is to be regarded as satisfactory. The instability leads to the fission of a planet into two main pieces of comparable though necessarily different masses, and it must be supposed that a number of small masses would stream out between them as they separated. The end members of this stream would be captured and become satellites of the adjacent planets, while the intermediate parts would escape both the components and thereafter become independent planets.

The process here envisaged is closely analogous to the mechanism of the original tidal theory of the origin of the solar system, but with the important difference and added advantage that in the present case the separation of the two main masses involves an actual break-up of a single mass across some layer of itself. The material near this layer will clearly be in an unstable region under the attractions of the two components, and will therefore be well situated for the formation of small bodies as the two masses fly apart. The details of the process do not admit of any mathematical treatment, but it obviously would be at least as efficacious as purely tidal action or even collisional forces. Furthermore, the material of the planet being liquid at this stage, the continued existence of bodies as small as the Moon, for example, does not present any difficulty.

Now if a rotating fluid mass breaks into a number of smaller parts in this way it is evident that, apart from effects of possible changes of density, these masses will have rotations comparable with that of the whole mass at the time of disruption. Each of the small masses retains the angular momentum it originally had. An exactly similar conclusion was advanced with some detail by Jeffreys † in connection with a criticism of the formation of the planets by purely tidal action. Rotation cannot be created or destroyed in a fluid by gravitational forces. Accordingly all planets formed by the fission of a large primitive planet must be regarded as being endowed with rapid rotations, and it has been shown by the author in the paper already referred to that the rotation periods would be of the order of a few hours. So that the fissional theory of the origin of satellites and the terrestrial planets ascribes initially rapid rotations to all these bodies. Some account must

* *M.N.*, 98, 633, 1938.

† *M.N.*, 89, 636, 1929.

therefore be given of how the terrestrial planets have come to possess smaller angular velocities.

It is now understood that the Moon is receding from the Earth under the action of tidal friction, and it is possible that the Moon was formerly a few radii of the Earth distant from it, and the latter would then necessarily have been rotating considerably more rapidly than at present. Here the reduction of the rate of rotation of the Earth depends chiefly on the presence of its comparatively massive satellite, the retarding effects of which are even now greater than those of the Sun. But the Earth now has the shortest rotation period of the four terrestrial planets, which seems to be contrary to what would be expected. It appears therefore that Mercury, Venus and probably Mars must in the past have been subjected to tidal influences at least as strong as those at present affecting the Earth's rotation. This view leads to some interesting consequences.

1. *Mercury and Venus*.—It has been shown by Jeffreys* that solar tidal friction could not by itself have lengthened the rotation period of Venus to its present value (> 20 days) if Venus ever had a short rotation period. The solar tides could produce on Venus about twice the present rate of retardation of the Earth, but this rate is small compared with the value when the Moon was considerably nearer. Venus must therefore have been affected by another body apart from the Sun. The hypothesis that Mercury was once a satellite of Venus and was driven beyond the region of stable satellite orbits by tidal friction appears to be the only possible suggestion that is consistent with the introduction of the necessary tidal effects. This proposal was investigated and discarded by Jeffreys† in connection with the resonance theory of the origin of satellites. But the argument that led him to abandon the hypothesis does not appear to be valid, as will shortly be shown.

If Venus rotated in a few hours and had a satellite as massive as Mercury, it is evident that, on the assumption of conditions for tidal friction similar to terrestrial ones, the evolution of the orbit of Mercury about Venus would proceed much more rapidly than the lunar orbit. Now the angular momentum of the Earth-Moon system is sufficient to increase the period of the Moon in its orbit to forty-seven days, and it may reasonably be conjectured that Venus would be capable of expelling Mercury to a distance corresponding to a similar period, though perhaps somewhat less, since Mercury is more massive than the Moon.

In discussing this question Jeffreys regarded Mercury as moving at any instant in a variational orbit about Venus under the combined action of the planet and the Sun, thereby ignoring any eccentricity in either the planetary orbit or the satellite orbit. This procedure is clearly not permissible in a general discussion of the problem, for the stability of the satellite orbit must involve the eccentricities. Moreover, these eccentricities cannot even be regarded as small, for when tidal friction operates to increase the distance of a satellite it in general also increases the orbital eccentricity.‡

* *The Earth*, p. 275, 1929.

† *M.N.*, 78, 439, 1918.

‡ Darwin, *Collected Works*, 2, 338.

Thus Mercury might come to move in a highly eccentric orbit and the stability become endangered more readily. In addition, if there exists some agency in the solar system that could reduce the eccentricity of the orbit of Mercury, as has been maintained by Jeffreys and others, it would be natural to suppose that the orbit of Venus could be similarly affected, and the assumption of zero eccentricity for the orbit of Venus can no longer be justified.

Jeffreys' investigation, restricted to zero eccentricities, showed that Mercury would begin its independent existence in an orbit of less eccentricity than the present value. But it is clear that the presence of eccentricity in the orbits of Venus and the satellite would permit the escaping body to have considerable range of velocity both as regards magnitude and direction. In addition to this the initial planetary orbit of Mercury would pass very near the orbit of Venus, and hence a close approach of the two bodies could occur and possibly give rise to considerable eccentricity in the orbit of Mercury even if its initial orbit were not far from circular. It appears therefore that the hypothesis that Mercury was once a satellite of Venus cannot be dismissed on dynamical grounds, while it is strongly supported by the present slow rotation of Venus.

Furthermore, during its existence as a satellite of Venus, Mercury would come to rotate in the same period as its orbital period about the planet, just as the Moon always faces the Earth. Therefore when Mercury became an independent planet it would have a long rotation period, probably about forty days, and the bodily tide produced by the Sun might alone be sufficient to cause it eventually to face the Sun.

2. *Mars*.—If a planet possesses several satellites their retarding effects are proportional to the sixth power of the angular diameters as seen from the centre of the primary. Estimates of the brightness of Phobos, the inner satellite of Mars, indicate that it has an angular diameter similar to that of the Sun, and therefore on account of its greater mean density must cause stronger tidal effects on Mars.* But the orbital period of Phobos is shorter than the rotation period of the planet, and hence tidal friction would cause Phobos to approach Mars. This satellite cannot therefore have much affected the rotation of the planet; for any change in the angular momentum of the planet must reappear as orbital angular momentum of the satellite, and any appreciable change of rotation of Mars would result in Phobos being brought within Roche's limit or even on to the surface of the planet. Moreover, the action of Phobos would be to increase the rate of rotation of the planet, whereas the search is for some process that can decrease it.

If therefore Mars once had a rapid rotation period, Phobos must be regarded as a satellite acquired recently in the history of the solar system; and as has been proposed for the outer satellites of Jupiter and Saturn, it may be surmised that both Phobos and Deimos are probably captured asteroids. The solar tides then become an available agency for slowing the rotation of Mars during its whole history prior to the capture of Phobos. Shallow seas may well have existed on Mars in the past, and

* *The Earth*, p. 275, 1929.

would have been capable of producing effects comparable with or even greater than the corresponding effects existing on the Earth.

3. *The Asteroids*.—The rotation period of several asteroids have been inferred from observed periodic fluctuations in brightness that can only be explained as being due to rotation. The consequent changes of brightness may result either from irregularity of form of the asteroid or to differences of reflecting power at various parts of its surface. The former cause has been well established for a number of asteroids, notably Eros. All the known rotation periods are short, *e.g.* Iris $6^h\ 12^m$, Eros $5^h\ 17^m$, Vesta $5^h\ 55^m$, Tercidina $8^h\ 47^m$.

Now it is generally agreed that the asteroids have resulted from the disruption of a single mass, and their irregular shapes suggest that at least part of the mass must have been solid at the time. It must also have been rotating at a similar rate to the observed rotations of asteroids. This last consideration indicates that the minor planets are more likely to have been formed from an independent planet than from a former satellite of any of the present planets; for it is well known that most satellites have slow rotations, since they always turn the same face towards their primary, and this effect was probably produced very early in the development of the solar system since the tidal effects of primaries on their satellites are so great.

The methods by which a planet can have been disrupted are discussed by Jeffreys in *The Earth*. The suggestion of the tidal action of Jupiter during a close encounter of the two planets appears to be the most acceptable. Though, as Jeffreys has pointed out, the explosion of a single mass is also a possible explanation, there is little to recommend it. The very nature of the asteroids points strongly to their having had a catastrophic origin, and if this is so it is not to be expected that a complete account of them can be given until their orbits, compositions, etc., are far better understood than at present.

4. *The Four Great Planets*.—The rotation periods of the outer planets of the solar system cannot have been much affected by the tides of their satellites, except perhaps in the case of Neptune. For if the satellites along with their orbital angular momenta were returned to their primary, in no instance, excepting Neptune, would any appreciable difference of rotation period occur. But it does not follow that the orbits of the satellites have been unaffected by these tidal influences. Indeed, there appear to be features of the satellite system that can only be explained in terms of the results of tidal friction.

It has been suggested by Jeffreys * that the distances of the nearer satellites have been reduced to the existing values by the action of the resisting medium. But it has been shown by the writer in a previous paper † that these satellites even now possess more angular momentum than they could derive during the process of fission that produced them and their primaries. Accordingly, if a resisting medium existed and operated to reduce the distances of the satellites, some other mechanism must have been available to overcome this and produce an increase of the mean distances.

* *The Earth*, p. 57.

† *M.N.*, 96, 566, 1936.

Apart from chance encounters with other planets, which would not only be rare but also be equally likely to produce an instantaneous decrease or increase, the only known cause is tidal friction. At present, however, tidal friction is almost completely absent from Jupiter * and presumably from the other great planets, but it is unlikely that this has always been the case. It is entirely possible that while these planets were only partly solidified at their surfaces the conditions may have led to considerable tidal lags and, even if existing for only a comparatively short time, the actual amount of tidal friction might have been great.

5. *Saturn's Rings*.—It is generally accepted that the particles of the rings must at one time have formed at least part of a single larger body, presumably a satellite, that was originally outside of the Roche limit but was brought within it. In discussing the effect of the resisting medium to reduce mean distances, Jeffreys has therefore suggested that the existence of Saturn's ring system is confirmatory evidence of the process. It can readily be shown, however, that tidal friction is equally capable of producing the necessary reduction of distance.

A satellite will approach its primary owing to tidal action if its orbital period is less than the rotation period of the planet. For Saturn, at a mean distance of 1.83 equatorial radii, a satellite would have the same rotation period as the planet. And it would not be disrupted if its density (supposed uniform) was such that the Roche limit was less than this, *i.e.* if

$$2.46 \text{ (Density of Saturn/Density of satellite)}^{\frac{1}{3}} < 1.83,$$

which requires the density of the satellite to exceed 1.78. Thus a satellite whose central parts had a density greater than 1.78 could remain intact at a mean distance corresponding to an orbital period less than the rotation period of Saturn. Tidal friction would then cause it to approach the planet and its stability eventually to be endangered.

If it is assumed that the thickness and distribution of the material of the ring are everywhere the same in it, the average angular momentum of the ring can be found, and accordingly an estimate of the distance of the satellite at the time of its disruption is obtained. In terms of the radius of the planet, the ring lies entirely between 1.2 and 2.4 radii and if it were collected into a single body the angular momentum would correspond to motion in a circular orbit at a distance of 1.85 radii. On the other hand, the main mass of the ring, as inferred from the brightness, lies chiefly between 1.2 and 2.2 radii † and the corresponding angular momentum is attained at a distance of 1.74 radii. But in addition to this, owing to collisions between the particles, the ring must always be losing matter possessing small angular momentum to the planet, and this must result in an increase in the average angular momentum. Hence the values obtained above for the distance of the original satellite must exceed the actual mean distance at the time it became disrupted. A satellite of density similar to that of the Moon could therefore have an orbital period less than the rotation period of Saturn and

* *M.N.*, 91, 706, 1931.

† Russell, Dugan and Stewart, *Astronomy*, p. 384 *et seq.*

thus be made to approach the planet by tidal friction, ultimately leading to disruption and the formation of a ring. Such a ring would tend to expand outwards as it approached a steady state.

6. *Uranus*.—The satellites of Uranus are also at distances requiring more angular momentum than the theory of their origin implies. But the satellites may be as large as 1000 miles in diameter, so that tidal friction could readily bring about the necessary increase of the mean distances.

The direction of the rotation axis of the planet itself calls for some comment however. Indeed, it appears at first sight to be one of the most extraordinary features of the solar system, now that the rotation of Neptune has been established as direct. Now, in the theory of the origin of the planets, their rotations are ascribed to the effects of turbulent motion during the collision of the parent stars. The planetary material that streams out between the stars would be endowed with rotation in such a way that any part of it would rotate about an axis perpendicular to the direction of motion of the colliding stars. But there is no reason why all parts of the material removed from the stars should rotate about parallel axes, and accordingly the material need not necessarily condense into planets rotating about parallel axes.

Again, the solar precession of a planet takes place in such a way that the axis of rotation maintains approximately a constant angle with the orbital plane, provided the latter remains fixed. But the inclinations of the planetary orbits to the invariable plane may have been much greater at one time than they are now (as is probably true also of the eccentricities), and hence even if two planets originally rotated about parallel axes there would be no reason for them to continue to do so.

Accordingly, the present direction of the axis of Uranus may be due to the combined influence of these two processes: first, a primitive rotation axis somewhat inclined to that of the other planets; and, second, the cumulative effects of precession as the orbital plane varied. It is not suggested that the orbital plane of Uranus has moved through 90° or thereabouts. A change of 20° in the inclination of the orbital plane could bring about a change of 40° in the inclination of the rotation axis to the perpendicular to the original plane. This is equally true of the other planets, so that if the inclination of the planetary orbits were originally of the order of 20° , the axes of rotation of the planets may have been approximately parallel. Thus the peculiar direction of the axis of Uranus may be explicable in dynamical terms.

7. *Neptune*.—In a previous paper * it has been shown, on the assumption of tidal friction on Neptune, that the retrograde motion of the satellite and the high eccentricity of the orbit of Pluto can both be explained. The suggestion advanced is that the mean distances of two originally direct satellites may have approached equality, eventually bringing about a close encounter between them. Such an encounter could succeed not only in ejecting one of the satellites altogether, but also in reversing the general direction of motion about the primary of the other one. The mass of Triton † is about four times that of the Moon, but it is not known with

* *M.N.*, 97, 108, 1936.

† *P.A.S.P.*, 43, 261, 1931.

much certainty. Judging from its brightness, Pluto must have a similar mass and may even be larger. Hence as satellites these bodies could produce considerable decrease in the rotation of Neptune, for the present orbital momentum of Triton is about 40 per cent. of the rotational momentum of the planet. It is noteworthy in this connection that the rotation period of Neptune, as far as can be ascertained,* is easily the longest of any of the great planets.

Further evidence of the existence of tidal friction on Neptune, at some time during its history, is provided by the actual orbit of the satellite, the eccentricity of which is unobservably small. Now if tidal friction operates to decrease the mean distance of a satellite it also decreases the eccentricity. It is highly improbable that the encounter that reversed the motion of Triton threw it into a circular orbit from the start; but the tidal friction would continue to act and the orbit would become more and more circular.

Summary.—In this paper some of the more detailed structure of the solar system is considered and, while some of the questions raised await complete investigation, it appears that the features under discussion are due to understood dynamical effects. The fission theory of the origin of the satellites and the terrestrial planets assigns rapid rotations to all these bodies in their initial stages. The knowledge of the lunar orbit indicates that formerly the Earth may have rotated in a few hours. In the case of Venus, however, the present rotation is so slow that sufficient tidal friction effects can only be obtained by reverting to the suggestion that Mercury was formerly a satellite of Venus. And this hypothesis appears to be admissible from a dynamical point of view. The slow rotation of Mars can have been produced by the solar tides only if its satellites are comparatively recent acquisitions. The satellite systems of the great planets all show that tidal friction has at some time produced a general increase of the mean distances. The same process could also have caused a satellite of short orbital period to approach Saturn, so that it would eventually be disrupted to give rise to the ring system. The exceptional direction of the rotation axis of Uranus may be partly primitive and partly due to precessional effects as the orbital plane varied. The slower rotation of Neptune, as compared with the other three great planets, and the peculiarities of the orbits of Triton and Pluto can also be accounted for in terms of the action of tidal friction.

* *P.A.S.P.*, 40, 234, 1928.

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A THEOREM ON THE PRESSURE IN THE INTERIOR OF A STAR.

W. Gleissberg.

(Communicated by Professor E. A. Milne)

Professor E. A. Milne has established the following theorem concerning the pressure in the interior of a star:—*

In any equilibrium configuration in which $\bar{\rho}$, the mean density inside r , always exceeds ρ , the actual density at r , the function $P + \frac{3}{8\pi} \frac{GM^2(r)}{r^4}$ decreases outwards.

Here r denotes the distance from the centre of the equilibrium configuration, P the total pressure at r , $M(r)$ the mass inside r , and G the constant of gravitation. Since

$$\frac{d\bar{\rho}}{dr} = \frac{3}{r}(\rho - \bar{\rho}),$$

the supposition that $\bar{\rho}$ always exceeds ρ is equivalent to the supposition that $\bar{\rho}$ decreases outwards; and as we have

$$\frac{3}{8\pi} \frac{GM^2(r)}{r^4} = \frac{1}{2} G \left(\frac{4}{3}\pi \right)^{\frac{1}{2}} \bar{\rho}^{\frac{1}{2}} M^{\frac{3}{2}}(r),$$

Milne's theorem states that in any configuration satisfying the condition of mechanical equilibrium, in which $\bar{\rho}$ decreases outwards, the function

$$P + \frac{1}{2} G \left(\frac{4}{3}\pi \right)^{\frac{1}{2}} \bar{\rho}^{\frac{1}{2}} M^{\frac{3}{2}}(r)$$

decreases outwards. On the other hand, it can be easily shown that, under the same conditions, the function

$$P + \frac{1}{2} G \left(\frac{4}{3}\pi \right)^{\frac{1}{2}} \rho_c^{\frac{1}{2}} M^{\frac{3}{2}}(r)$$

increases outwards. Here and in the following, quantities marked with the index c belong to the centre of the configuration.

A similar theorem has been recently published by the writer:—†

In any configuration satisfying the condition of mechanical equilibrium, in which $\bar{\rho}$ decreases outwards, the function

$$(P - \bar{P})M(r) + \frac{1}{5} G \left(\frac{4}{3}\pi \right)^{\frac{1}{2}} \bar{\rho}^{\frac{1}{2}} M^{\frac{5}{2}}(r)$$

decreases outwards and the function

$$(P - \bar{P})M(r) + \frac{1}{5} G \left(\frac{4}{3}\pi \right)^{\frac{1}{2}} \rho_c^{\frac{1}{2}} M^{\frac{5}{2}}(r)$$

increases outwards.

* *M.N.*, 96, 179, 1936, Theorem II.

† *A.N.*, 267, 121, 1938, Theorem II.

Here the mean total pressure $\bar{P}$, inside r , is defined by the equation

$$\bar{P} = \frac{1}{M(r)} \int_0^{M(r)} P dM(r). \quad (1)$$

In the present paper I prove a more general theorem which includes Milne's and the writer's theorems mentioned above. This general theorem refers to the generalized mean total pressure $\bar{P}^{(n)}$ defined for all integers $n \geq 0$ by the equations

$$\bar{P}^{(n)} = \begin{cases} P_c, & \text{if } n = 0 \\ \frac{1}{M^n(r)} \int_0^{M^n(r)} P d(M^n(r)), & \text{if } n = 1, 2, 3, \dots \text{ in inf.} \end{cases} \quad (2)$$

Theorem.—In any configuration satisfying the condition of mechanical equilibrium, in which $\bar{\rho}$ decreases outwards, the function

$$(P - \bar{P}^{(n)})M^n(r) + \frac{G}{3n+2} \left(\frac{4}{3}\pi\right)^{\frac{1}{2}} \bar{\rho}^{\frac{1}{2}} M^{n+\frac{3}{2}}(r)$$

decreases outwards and the functions

$$(P - \bar{P}^{(n)})M^n(r) + \frac{G}{3n+2} \left(\frac{4}{3}\pi\right)^{\frac{1}{2}} \rho_c^{\frac{1}{2}} M^{n+\frac{3}{2}}(r)$$

and

$$[3P + (3n+2)\bar{P}^{(n)} - (3n+5)\bar{P}^{(n+1)}]M^{n+1}(r)$$

increase outwards.

Proof.—If we set

$$(P - \bar{P}^{(n)})M^n(r) + \frac{G}{3n+2} \left(\frac{4}{3}\pi\right)^{\frac{1}{2}} \bar{\rho}^{\frac{1}{2}} M^{n+\frac{3}{2}}(r) = \chi^{(n)}, \quad (3)$$

we have

$$\frac{d\chi^{(n)}}{dM(r)} = \frac{dP}{dM(r)} M^n(r) + nPM^{n-1}(r) - \frac{d[\bar{P}^{(n)}M^n(r)]}{dM(r)} + \frac{4G}{9n+6} \left(\frac{4}{3}\pi\right)^{\frac{1}{2}} \bar{\rho}^{\frac{1}{2}} M^{n+\frac{3}{2}}(r) \frac{d\bar{\rho}}{dM(r)} + \frac{1}{3}G \left(\frac{4}{3}\pi\right)^{\frac{1}{2}} \bar{\rho}^{\frac{1}{2}} M^{n-\frac{1}{2}}(r). \quad (4)$$

The condition of mechanical equilibrium can be put into the following form :— *

$$\frac{dP}{dM(r)} = -\frac{1}{3}G \left(\frac{4}{3}\pi\right)^{\frac{1}{2}} \bar{\rho}^{\frac{1}{2}} M^{-\frac{1}{2}}(r), \quad (5)$$

and from the definition of $\bar{P}^{(n)}$ given by (2) we get

$$\frac{d[\bar{P}^{(n)}M^n(r)]}{dM(r)} = nPM^{n-1}(r). \quad (6)$$

Thus equation (4) reduces to

$$\frac{d\chi^{(n)}}{dM(r)} = \frac{4G}{9n+6} \left(\frac{4}{3}\pi\right)^{\frac{1}{2}} \bar{\rho}^{\frac{1}{2}} M^{n+\frac{3}{2}}(r) \frac{d\bar{\rho}}{dM(r)}. \quad (7)$$

* Cf. *A.N.*, 267, 123, eq. (9).

By hypothesis,

$$\frac{d\bar{\rho}}{dr} < 0,$$

and as $\frac{dM(r)}{dr}$ is always positive, we have

$$\frac{d\bar{\rho}}{dM(r)} < 0.$$

Thus we get from (7)

$$\frac{d\chi^{(n)}}{dM(r)} < 0 \quad \text{or} \quad \frac{d\chi^{(n)}}{dr} < 0,$$

which proves the first part of the theorem.

Setting

$$(P - \bar{P}^{(n)})M^n(r) + \frac{G}{3n+2} \left(\frac{4}{3}\pi\right)^{\frac{1}{2}} \rho_c^{\frac{3}{2}} M^{n+\frac{3}{2}}(r) = \psi^{(n)}, \quad (8)$$

we have

$$\frac{d\psi^{(n)}}{dM(r)} = \frac{dP}{dM(r)} M^n(r) + nPM^{n-1}(r) - \frac{d[\bar{P}^{(n)}M^n(r)]}{dM(r)} + \frac{1}{3}G\left(\frac{4}{3}\pi\right)^{\frac{1}{2}} \rho_c^{\frac{3}{2}} M^{n-\frac{1}{2}}(r).$$

Owing to (5) and (6) this gives

$$\frac{d\psi^{(n)}}{dM(r)} = \frac{1}{3}G\left(\frac{4}{3}\pi\right)^{\frac{1}{2}} M^{n-\frac{1}{2}}(r)(\rho_c^{\frac{3}{2}} - \bar{\rho}^{\frac{3}{2}}).$$

According to the supposition that $\bar{\rho}$ decreases outwards, we have, at all distances r from the centre ($r > 0$), $\rho_c > \bar{\rho}$; hence

$$\frac{d\psi^{(n)}}{dM(r)} > 0 \quad \text{or} \quad \frac{d\psi^{(n)}}{dr} > 0,$$

which proves the second part of the theorem.

From the first two parts of the theorem which have been hitherto proved we deduce an inequality by means of which its third part can be easily proved. By substituting $r=0$, we obtain from (3) and (8) *

$$\chi_c^{(n)} = \psi_c^{(n)} = 0. \quad (9)$$

Therefore, according to the first two parts of the theorem we have the inequality

$$\chi^{(n)} < 0 < \psi^{(n)} \quad (10)$$

at all distances r from the centre ($r > 0$). From (3), (8) and (10) we get

$$\frac{G}{3n+2} \left(\frac{4}{3}\pi\right)^{\frac{1}{2}} \bar{\rho}^{\frac{3}{2}} M^{\frac{3}{2}}(r) < \bar{P}^{(n)} - P < \frac{G}{3n+2} \left(\frac{4}{3}\pi\right)^{\frac{1}{2}} \rho_c^{\frac{3}{2}} M^{\frac{3}{2}}(r). \quad (11)$$

Now we can pass on to the proof of the third part of the theorem. Setting

$$[3P + (3n+2)\bar{P}^{(n)} - (3n+5)\bar{P}^{(n+1)}]M^{n+1}(r) = \omega^{(n)}, \quad (12)$$

* As, by definition, $\bar{P}^{(0)} = P_c$, equation (9) is true also in case $n=0$.

we have

$$\frac{d\omega^{(n)}}{dM(r)} = 3 \frac{dP}{dM(r)} M^{n+1}(r) + 3(n+1)PM^n(r) + (3n+2) \frac{d[\bar{P}^{(n)}M^n(r)]}{dM(r)} M(r) \\ + (3n+2)\bar{P}^{(n)}M^n(r) - (3n+5) \frac{d[\bar{P}^{(n+1)}M^{n+1}(r)]}{dM(r)}. \quad (13)$$

Substituting, in (6), $n+1$ for n , we have

$$\frac{d[\bar{P}^{(n+1)}M^{n+1}(r)]}{dM(r)} = (n+1)PM^n(r). \quad (14)$$

Substituting (5), (6) and (14) in (13), we find after rearrangement

$$\frac{d\omega^{(n)}}{dM(r)} = (3n+2)M^n(r) \left[\bar{P}^{(n)} - P - \frac{G}{3n+2} \left(\frac{4}{3}\pi \right)^{\frac{1}{2}} \bar{\rho}^{\frac{1}{2}} M^{\frac{2}{3}}(r) \right].$$

From (11) it follows that the right-hand side of this equation is always positive. Hence

$$\frac{d\omega^{(n)}}{dM(r)} > 0 \quad \text{or} \quad \frac{d\omega^{(n)}}{dr} > 0.$$

Thus the third part of the theorem has also been proved.

Applying (12) to the centre of the configuration, we have

$$\omega_c^{(n)} = 0.$$

Hence it follows from the third part of the theorem that, at all distances r from the centre ($r > 0$),

$$\omega^{(n)} > 0. \quad (15)$$

From (12) and (15) we have

$$3P + (3n+2)\bar{P}^{(n)} - (3n+5)\bar{P}^{(n+1)} > 0,$$

or

$$(3n+2)(\bar{P}^{(n)} - P) > (3n+5)(\bar{P}^{(n+1)} - P). \quad (16)$$

On the other hand, inequality (11) can be written in the following form:—

$$G\left(\frac{4}{3}\pi\right)^{\frac{1}{2}} \rho_c^{\frac{1}{2}} M^{\frac{2}{3}}(r) > (3n+2)(\bar{P}^{(n)} - P) > G\left(\frac{4}{3}\pi\right)^{\frac{1}{2}} \bar{\rho}^{\frac{1}{2}} M^{\frac{2}{3}}(r). \quad (17)$$

It may be noticed that the outer members of this inequality are independent of n . Therefore, if we substitute $n+1$ instead of n , we get from (17)

$$G\left(\frac{4}{3}\pi\right)^{\frac{1}{2}} \rho_c^{\frac{1}{2}} M^{\frac{2}{3}}(r) > (3n+5)(\bar{P}^{(n+1)} - P) > G\left(\frac{4}{3}\pi\right)^{\frac{1}{2}} \bar{\rho}^{\frac{1}{2}} M^{\frac{2}{3}}(r). \quad (18)$$

Combining (16), (17) and (18) we find

$$G\left(\frac{4}{3}\pi\right)^{\frac{1}{2}} \rho_c^{\frac{1}{2}} M^{\frac{2}{3}}(r) > (3n+2)(\bar{P}^{(n)} - P) > (3n+5)(\bar{P}^{(n+1)} - P) > G\left(\frac{4}{3}\pi\right)^{\frac{1}{2}} \bar{\rho}^{\frac{1}{2}} M^{\frac{2}{3}}(r).$$

In this inequality the values 0, 1, 2, 3, etc. can be substituted for n ; thus we have deduced from the theorem the following corollary:—

Corollary I.—In any configuration satisfying the condition of mechanical equilibrium, in which $\bar{\rho}$ decreases outwards, we have, at all distances r from the centre ($r > 0$), the following infinite series of inequalities:—

$$G\left(\frac{4}{3}\pi\right)^{\frac{1}{2}} \rho_c^{\frac{1}{2}} M^{\frac{2}{3}}(r) > 2(P_c - P) > 5(\bar{P}^{(1)} - P) > 8(\bar{P}^{(2)} - P) > 11(\bar{P}^{(3)} - P) \\ > \dots > G\left(\frac{4}{3}\pi\right)^{\frac{1}{2}} \bar{\rho}^{\frac{1}{2}} M^{\frac{2}{3}}(r).$$

The condition of radiative equilibrium can be written in the following form :— *

$$\frac{dp_r}{dM(r)} = -\frac{1}{3c} \left(\frac{4}{3}\pi\right)^{\frac{1}{2}} \bar{\rho}^{\frac{4}{3}} k \bar{\epsilon} M^{-\frac{1}{3}}(r), \quad (19)$$

where p_r denotes the radiation pressure, k the absorption coefficient, $4\pi\bar{\epsilon}$ the mean amount of energy produced inside r in one second by unit mass, and c the velocity of light. This equation can be obtained formally from (5) by substituting p_r for P , $\bar{\rho}^{\frac{4}{3}} k \bar{\epsilon}$ for $\bar{\rho}^{\frac{4}{3}}$ and $\frac{1}{c}$ for G . Hence it is clear that the following corollaries can be deduced from the theorem and from its Corollary I.

Corollary II.—In any configuration satisfying the condition of radiative equilibrium, in which $\bar{\rho}^{\frac{4}{3}} k \bar{\epsilon}$ decreases outwards, the function

$$(p_r - \bar{p}_r^{(n)})M^n(r) + \frac{1}{(3n+2)c} \left(\frac{4}{3}\pi\right)^{\frac{1}{2}} \bar{\rho}^{\frac{4}{3}} k \bar{\epsilon} M^{n+\frac{2}{3}}(r)$$

decreases outwards and the functions

$$(p_r - \bar{p}_r^{(n)})M^n(r) + \frac{1}{(3n+2)c} \left(\frac{4}{3}\pi\right)^{\frac{1}{2}} \rho_c^{\frac{4}{3}} (k\bar{\epsilon})_c M^{n+\frac{2}{3}}(r)$$

and

$$[3p_r + (3n+2)\bar{p}_r^{(n)} - (3n+5)\bar{p}_r^{(n+1)}]M^{n+1}(r)$$

increase outwards.

Corollary III.—In any configuration satisfying the condition of radiative equilibrium, in which $\bar{\rho}^{\frac{4}{3}} k \bar{\epsilon}$ decreases outwards, we have, at all distances r from the centre ($r > 0$), the following infinite series of inequalities :—

$$\begin{aligned} \frac{1}{c} \left(\frac{4}{3}\pi\right)^{\frac{1}{2}} \rho_c^{\frac{4}{3}} (k\bar{\epsilon})_c M^{\frac{2}{3}}(r) &> 2[(p_r)_c - p_r] > 5(\bar{p}_r^{(1)} - p_r) > 8(\bar{p}_r^{(2)} - p_r) > 11(\bar{p}_r^{(3)} - p_r) \\ &> \dots > \frac{1}{c} \left(\frac{4}{3}\pi\right)^{\frac{1}{2}} \bar{\rho}^{\frac{4}{3}} k \bar{\epsilon} M^{\frac{2}{3}}(r). \end{aligned}$$

Here the generalized mean radiation pressure $\bar{p}_r^{(n)}$ is defined by the equations

$$\bar{p}_r^{(n)} = \begin{cases} (p_r)_c, & \text{if } n=0 \\ \frac{1}{M^n(r)} \int_0^{M^n(r)} p_r d(M^n(r)), & \text{if } n=1, 2, 3, \dots \text{ in inf.} \end{cases}$$

Subtracting the condition (19) of radiative equilibrium from the condition (5) of mechanical equilibrium and using the equation

$$p_g = P - p_r,$$

where p_g denotes the gas pressure, we find

$$\frac{dp_g}{dM(r)} = -\frac{1}{3c} \left(\frac{4}{3}\pi\right)^{\frac{1}{2}} \bar{\rho}^{\frac{4}{3}} (cG - k\bar{\epsilon}) M^{-\frac{1}{3}}(r).$$

* Cf. *A.N.*, 267, 124, eq. (15).

As this equation can be obtained formally by substituting, in (5), p_g for P , $\bar{\rho}^{\frac{1}{3}}(cG - k\bar{\epsilon})$ for $\bar{\rho}^{\frac{1}{3}}$ and $\frac{1}{c}$ for G , the following corollaries can also be deduced from the theorem and from its Corollary I, provided that $cG - k\bar{\epsilon} > 0$ at all distances from the centre.

Corollary IV.—In any configuration satisfying the conditions of mechanical and radiative equilibrium, in which $cG - k\bar{\epsilon}$ is always positive and $\bar{\rho}^{\frac{1}{3}}(cG - k\bar{\epsilon})$ decreases outwards, the function

$$(p_g - \bar{p}_g^{(n)})M^n(r) + \frac{1}{(3n+2)c} \left(\frac{4}{3}\pi\right)^{\frac{1}{3}} \bar{\rho}^{\frac{1}{3}}(cG - k\bar{\epsilon})M^{n+\frac{2}{3}}(r)$$

decreases outwards and the functions

$$(p_g - \bar{p}_g^{(n)})M^n(r) + \frac{1}{(3n+2)c} \left(\frac{4}{3}\pi\right)^{\frac{1}{3}} \rho_c^{\frac{1}{3}}[cG - (k\bar{\epsilon})_c]M^{n+\frac{2}{3}}(r)$$

and

$$[3p_g + (3n+2)\bar{p}_g^{(n)} - (3n+5)\bar{p}_g^{(n+1)}]M^{n+1}(r)$$

increase outwards.

Corollary V.—In any configuration satisfying the conditions of mechanical and radiative equilibrium, in which $cG - k\bar{\epsilon}$ is always positive and $\bar{\rho}^{\frac{1}{3}}(cG - k\bar{\epsilon})$ decreases outwards, we have, at all distances r from the centre ($r > 0$), the following infinite series of inequalities:—

$$\begin{aligned} \frac{1}{c} \left(\frac{4}{3}\pi\right)^{\frac{1}{3}} \rho_c^{\frac{1}{3}}[cG - (k\bar{\epsilon})_c]M^{\frac{2}{3}}(r) &> 2[(p_g)_c - p_g] > 5(\bar{p}_g^{(1)} - p_g) > 8(\bar{p}_g^{(2)} - p_g) \\ &> 11(\bar{p}_g^{(3)} - p_g) > \dots > \frac{1}{c} \left(\frac{4}{3}\pi\right)^{\frac{1}{3}} \bar{\rho}^{\frac{1}{3}}(cG - k\bar{\epsilon})M^{\frac{2}{3}}(r). \end{aligned}$$

Here the generalized mean gas pressure $\bar{p}_g^{(n)}$ is defined by

$$\bar{p}_g^{(n)} = \begin{cases} (p_g)_c, & \text{if } n=0 \\ \frac{1}{M^n(r)} \int_0^{M^n(r)} p_g d(M^n(r)), & \text{if } n=1, 2, 3, \dots \text{ in inf.} \end{cases}$$

Finally, let us consider two special cases, $n=0$ and $n=1$, of the theorem and of its corollaries.

The Case $n=0$.—In this case the first part of the theorem states that in any equilibrium configuration in which $\bar{\rho}$ decreases outwards, the function

$$P - P_c + \frac{1}{2}G \left(\frac{4}{3}\pi\right)^{\frac{1}{3}} \bar{\rho}^{\frac{1}{3}}M^{\frac{2}{3}}(r)$$

decreases outwards. Since P_c is a constant, this yields Milne's theorem mentioned at the beginning of this paper.

Omitting, in the inequalities of Corollary I, all members which relate to values of n above 0, we have

$$\frac{1}{2}G \left(\frac{4}{3}\pi\right)^{\frac{1}{3}} \rho_c^{\frac{1}{3}}M^{\frac{2}{3}}(r) > P_c - P > \frac{1}{2}G \left(\frac{4}{3}\pi\right)^{\frac{1}{3}} \bar{\rho}^{\frac{1}{3}}M^{\frac{2}{3}}(r);$$

this yields a theorem established by Chandrasekhar.*

* *M.N.*, 96, 644, 1936, Theorem 1.

In case $n=0$ the first two parts of Corollary II state that in any equilibrium configuration in which $\bar{\rho}^{\frac{1}{3}}k\bar{\epsilon}$ decreases outwards, the function

$$p_r - (p_r)_c + \frac{1}{2c} \left(\frac{4}{3}\pi\right)^{\frac{1}{3}} \bar{\rho}^{\frac{4}{3}} k \bar{\epsilon} M^{\frac{2}{3}}(r)$$

decreases outwards and the function

$$p_r - (p_r)_c + \frac{1}{2c} \left(\frac{4}{3}\pi\right)^{\frac{1}{3}} \rho_c^{\frac{4}{3}} (k\bar{\epsilon})_c M^{\frac{2}{3}}(r)$$

increases outwards. This yields a theorem proved by the writer in a previous paper.*

The Case $n=1$.—In this case the first two parts of the theorem state that in any equilibrium configuration in which $\bar{\rho}$ decreases outwards, the function

$$(P - \bar{P}^{(1)})M(r) + \frac{1}{5}G\left(\frac{4}{3}\pi\right)^{\frac{1}{3}} \bar{\rho}^{\frac{4}{3}} M^{\frac{2}{3}}(r)$$

decreases outwards and the function

$$(P - \bar{P}^{(1)})M(r) + \frac{1}{5}G\left(\frac{4}{3}\pi\right)^{\frac{1}{3}} \rho_c^{\frac{4}{3}} M^{\frac{2}{3}}(r)$$

increases outwards. Since $\bar{P}^{(1)}$ is identical with $\bar{P}$ as defined by (1), this yields a theorem of the writer mentioned in the introduction of the present paper.

Omitting, in the inequalities of Corollaries I and III, all members which relate to values of n higher than 1, we have

$$\frac{1}{5}G\left(\frac{4}{3}\pi\right)^{\frac{1}{3}} \rho_c^{\frac{4}{3}} M^{\frac{2}{3}}(r) > \frac{2}{5}(P_c - P) > \bar{P}^{(1)} - P > \frac{1}{5}G\left(\frac{4}{3}\pi\right)^{\frac{1}{3}} \bar{\rho}^{\frac{4}{3}} M^{\frac{2}{3}}(r)$$

and

$$\frac{1}{5c} \left(\frac{4}{3}\pi\right)^{\frac{1}{3}} \rho_c^{\frac{4}{3}} (k\bar{\epsilon})_c M^{\frac{2}{3}}(r) > \frac{2}{5}[(p_r)_c - p_r] > \bar{p}_r^{(1)} - p_r > \frac{1}{5c} \left(\frac{4}{3}\pi\right)^{\frac{1}{3}} \bar{\rho}^{\frac{4}{3}} k \bar{\epsilon} M^{\frac{2}{3}}(r).$$

These inequalities include two theorems proved by the writer in a previous paper.†

In case $n=1$ the first two parts of Corollary II state that in any equilibrium configuration in which $\bar{\rho}^{\frac{1}{3}}k\bar{\epsilon}$ decreases outwards, the function

$$(p_r - \bar{p}_r^{(1)})M(r) + \frac{1}{5c} \left(\frac{4}{3}\pi\right)^{\frac{1}{3}} \bar{\rho}^{\frac{4}{3}} k \bar{\epsilon} M^{\frac{2}{3}}(r)$$

decreases outwards and the function

$$(p_r - \bar{p}_r^{(1)})M(r) + \frac{1}{5c} \left(\frac{4}{3}\pi\right)^{\frac{1}{3}} \rho_c^{\frac{4}{3}} (k\bar{\epsilon})_c M^{\frac{2}{3}}(r)$$

increases outwards. This yields a theorem proved by the writer in a previous paper.‡

* *M.N.*, 98, 571, 1938, Auxiliary Theorem. The supposition made there, namely, that $\bar{\rho}$ and $\bar{\rho}^{\frac{1}{3}}k\bar{\epsilon}$ both decrease outwards, includes the supposition made here, namely, that $\bar{\rho}^{\frac{1}{3}}k\bar{\epsilon}$ decreases outwards.

† *A.N.*, 267, 121, 1938, Theorems 13 and 14.

‡ *Ibid.*, Theorem 12.

Summary.—In this paper I prove a theorem concerning the total pressure and the generalized mean total pressure in the interior of an equilibrium configuration, and from this theorem I deduce five corollaries concerning the total pressure, the radiation pressure, the gas pressure and their generalized mean values. It is shown that seven theorems published earlier by Milne, Chandrasekhar and the writer can be considered as special cases of the new theorem and of its corollaries discussed in the present paper.

*University Observatory,
Bayazit, Istanbul (Turkey) :
1938 December.*

THE TIDAL EFFECT ON THE VARIATION OF LATITUDE AT GREENWICH.

H. Spencer Jones, F.R.S., Astronomer Royal.

1. The theoretical value of the effect on the variation of latitude produced by the semi-diurnal lunar tide is given, on the equilibrium theory, by

$$\frac{3}{4} \frac{M}{E} \left(1 - \frac{5}{2} e^2 \right) \left(\frac{a}{c} \right)^3 \cos^4 \frac{I}{2} \sin 2\phi \cos 2t,$$

where E , M denote the masses of the Earth and Moon respectively, a is the radius of the Earth, c is the mean distance of the Moon, I is the inclination of the orbit of the Moon to the equator, t is the hour angle of the Moon.

From this expression it follows that the effect, for the latitude of Greenwich, is given by

$$0''.0077 \cos 2t.$$

In the derivation of this expression small effects arising from the ellipsoidal form of the Earth and from the variation of gravity with latitude have been neglected. It has also been assumed that there is no deformation of the Earth as a whole. To take account of elastic yielding, we can introduce a factor $(1 + \xi)$; the value of this factor for assumptions that represent the actual Earth is not known with certainty, but W. D. Lambert has estimated that its value probably lies between 1.1 and 1.2.

The full theoretical value may therefore be written in the form

$$0''.0077(1 + \xi) \cos 2t.$$

2. To investigate such an effect from the observations at Greenwich with the Cookson floating zenith telescope in the years 1911–1936, the residuals of the observed *minus* tabular zenith distances north for each group were used. The residuals from the tabular zenith distances computed with the final star places were corrected for (a) the latitude variations, read off from a smoothed curve representing the final values, from which wind effects had been eliminated; (b) the group corrections, using the final values from the twenty-five years' observations; (c) the wind effect, as derived in the definitive discussion of the 1911–1936 observations; (d) the diurnal variation of latitude, as derived in the same discussion.

The residuals so obtained were collected, according to the hour angle of the Moon at the time of observation, into 24 groups corresponding to hour angles 0^h to 1^h , 1^h to 2^h , etc.

Mean values were taken, residuals exceeding $0''.2$ being discarded because they were liable to produce an undue effect on the mean. The

corresponding groups 12 hours apart were combined and the following mean values were obtained :—

TABLE
Tidal Effect in the Latitude Variation at Greenwich

$2t$	Mean Residual (unit $0''.0001$)	Number of Plates
0		
15	- 69	404
45	+ 51	376
75	+ 56	352
105	- 22	358
135	+ 27	367
165	+ 5	377
195	+ 110	391
225	+ 104	391
255	+ 3	379
285	+ 11	381
315	- 16	387
345	- 80	390

The harmonic analysis of these figures, taken with equal weight, gives

$$+ 0''.0015 - ''\cdot 0050 \cos 2t - ''\cdot 0002 \sin 2t.$$

The lunar tidal effect in the variation of latitude at Greenwich is therefore $0''.0050 \cos (2t - 2^\circ\cdot 3)$.

The probable error of each entry in the table, derived from the residuals obtained after eliminating the above tidal effect, is $\pm ''\cdot 0033$. As each entry is the mean of about 380 plates, the inferred probable error for a single plate is $\pm ''\cdot 064$, which is satisfactory. The probable errors of the coefficients of $\cos 2t$ and $\sin 2t$ are $\pm ''\cdot 0013$.

The tidal effect, derived from the observations, may therefore be expressed in the form

$$(0''.0050 \pm ''\cdot 0013) \cos (2t - 2^\circ \pm 15^\circ)$$

as compared with the theoretical value

$$0''.0077(1 + \xi) \cos 2t.$$

3. The agreement in phase between the observed and computed effects is satisfactory, but the observed coefficient is smaller than the theoretical value. In the theoretical derivation the mean values of certain quantities relating to the Moon are derived by harmonic analysis on the assumption that the hour angle of the mean Moon is used. The observations have been analysed in relation to the hour angle of the true Moon; if the hour angle of the mean Moon had been used, the derived coefficient would have been somewhat smaller. It appears reasonably certain, therefore, that for Greenwich the observed value of the tidal effect in the latitude variation is smaller than the theoretical value.

It may be noted, incidentally, that so small a quantity cannot be determined with great percentage accuracy even from a total of more than 4500 plates. The present determination has, however, greater weight than any previous determination from observations of latitude variation.

4. There is a cause operating at Greenwich which reduces the theoretical tidal effect in the latitude variation observations. This is the rise and fall of the tide in the River Thames, to the north of the observatory. The observatory is situated a little to the south of the river, at the southern extremity of a large elbow. The tidal constant at Greenwich being $12^h 11^m$, the maxima and minima of the gravitational effects from the water in the river coincide closely in point of time with the maxima and minima of the general tidal effect on the variation of latitude.

A numerical estimate of the effects due to the tides in the river has been made. The mean difference in height between high and low tides at Greenwich is 20.91 feet at springs and 14.08 feet at neaps.* The mean of these two values was used to represent the average rise and fall of the river between high and low tides. The effect on the vertical at Greenwich of the attraction of the water between high and low tides was computed, using a 6-inch to the mile ordnance survey map for the data for the numerical integration. The total reflection in a northerly direction so obtained was $0''.0042$. The effect of the tides in the river can therefore be represented approximately by a term $0''.0021 \cos(2t - 5^\circ.5)$. This term must be added to the term deduced from the latitude variation observations to obtain the observational value of the general tidal effect.

5. It may therefore be concluded that the general tidal effect in the latitude variation at Greenwich is represented by

$$(0''.0071 \pm ''\cdot0013) \cos 2t + (0''.0004 \pm ''\cdot0013) \sin 2t \\ = (0''.0071 \pm ''\cdot0013) \cos(2t - 3^\circ \pm 15^\circ).$$

The probable errors of the coefficients of $\cos 2t$ and $\sin 2t$ have not been increased by the inclusion of the tidal effect due to the river, because the probable error of the latter effect is not likely to exceed $\pm 0''.0002$.

The observational value is now not greatly different from the theoretical value. A comparison between the two gives $(1 + \xi) = 0.92 \pm 0.17$; the discordance from the probable value of 1.1 to 1.2 is not of great significance. The observations of latitude variation cannot, unfortunately, provide a very strong determination of this factor.

Summary.—The lunar tidal effect on the latitude at Greenwich has been investigated from the observations for latitude variation made with the Cookson telescope from 1911 to 1936. The observed effect is $0''.0050 \cos(2t - 2^\circ.3)$. There is in addition a calculable effect due to the tides in the River Thames, estimated to amount to $0''.0021 \cos(2t - 5^\circ.5)$. The true lunar tidal effect in the observations is therefore

$$(0''.0071 \pm ''\cdot0013) \cos(2t - 3^\circ \pm 15^\circ),$$

which may be compared with the theoretical effect, $''\cdot0077(1 + \xi) \cos 2t$. The factor $(1 + \xi)$ depends on the deformation of the surface of the Earth and is estimated to be between 1.1 and 1.2. The difference between the observed and theoretical values is not of great significance.

* These figures, which refer to Greenwich Pier, were kindly supplied by the Port of London Authority.

A COMPARISON OF THE DECLINATIONS OF THE BOSS GENERAL CATALOGUE WITH THOSE DERIVED FROM THE OBSERVATIONS WITH THE COOKSON FLOATING TELESCOPE TO DETERMINE THE VARIATION OF THE LATITUDE AT GREENWICH DURING THE YEARS 1911- 1936.

H. R. Hulme, M.A., Ph.D.

(Communicated by the Astronomer Royal)

During the years 1911-1936 a continuous series of observations has been made with the Cookson Floating Telescope to determine the variation of the latitude at Greenwich. In the course of these observations the mean zenith distances (north) of 69 pairs of stars have been determined, for the epoch 1925.0, with a high degree of accuracy, and free from errors arising from an unknown variation of latitude. (A full discussion of the observations will be given in a forthcoming Greenwich publication.)

For these 69 pairs the mean zenith distances (north) given by the General Catalogue of Boss have been calculated; the differences, Boss - Cookson, were analysed with respect to variation with R.A. The value assumed for the latitude when calculating the mean zenith distances from the Boss declinations was $51^{\circ} 28' 38''.75$ as given on p. 45 of the Greenwich publication of 1921 on Latitude Variation. With this value we find for the mean zenith distance north for epoch 1925.0

$$\text{Boss} - \text{Cookson} = (12.8 \pm 1.8) \sin \theta + (5.4 \pm 1.8) \cos \theta + 29 \pm 1.3, \quad (1)$$

where θ is the mean R.A. of the pair and the unit is $0''.01$.

It is also possible to obtain the mean P.M. in declination of the two stars in each pair, apart from a constant term which is bound up with any secular change of latitude. For the P.M. the results are not so well determined.

We find

$$\text{Boss} - \text{Cookson (P.M.)} = (16 \pm 9) \sin \theta + (0 \pm 9) \cos \theta + (13 \pm 6), \quad (2)$$

the unit being $''\cdot 01$ per century.

The constant term in (1) is to be explained as follows. The latitude quoted was obtained (in 1918) by applying to 21 pairs of stars the equation

$$\text{Mean Declination} - \text{Latitude of Station} = \text{Mean Zenith distance north},$$

the right-hand side being an observed quantity. The declinations were taken from the Preliminary General Catalogue of Boss, which, however, requires two corrections:—

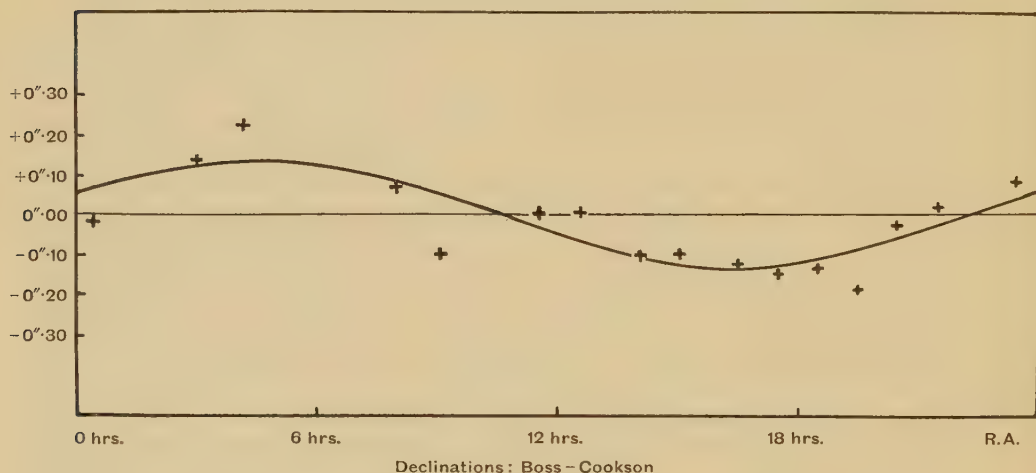
- (a) A correction of $+0''.23$ to bring it into accord with the G.C. This figure is taken from Boss G.C., vol. i, Table 23, and applies to a mean declination 51° ;
- (b) the periodic term given above, which we are to apply to both Boss catalogues. (We neglect the very uncertain correction to the P.M.)

The correction (b) would have no effect on the mean declination of the 21 pairs if they were evenly distributed in R.A., but 17 of them have a R.A. between 12 and 24 hours, where the Boss values are to be increased. This introduces an extra correction amounting to $0''.06$, so that the average value of the mean declinations is to be increased by $0''.29$, when the corrected latitude becomes $51^\circ 28' 39''.04$. Using this value we then have, for epoch 1925.0,

$$\text{Boss} - \text{Cookson} = 12.8 \sin \theta + 5.4 \cos \theta, \quad (3)$$

the unit being $''\cdot01$.

The results for the 16 groups into which the pairs are divided and the curve given by (3) are shown below.



We have also compared the Boss declinations with the FK_3 declinations for a selection of stars with declinations 34° – 70° . For each hour of R.A. six stars were chosen, so that the mean declination was about 51° and the distribution was roughly the same as the Cookson stars. We find

$$\text{Boss} - FK_3 = 12.9 \sin \theta + 1.4 \cos \theta - 14.7 \quad (4)$$

for epoch 1950.0, and

$$\text{Boss} - FK_3 = 9.3 \sin \theta + 2.9 \cos \theta - 9.9 \quad (5)$$

for epoch 1925.0, the unit being $''\cdot01$.

Comparing (3) and (5) we see that there is reason to believe that the declinations of the General Catalogue (between $+34^\circ$ and $+70^\circ$) require a correction depending on right ascension of the magnitude suggested in equation (1).

It should be mentioned that the absence of any constant term in equation (3) does not imply that the Greenwich observations confirm the *mean* value of the Boss declinations rather than the *FK*₃, as the latitude assumed in deriving (3) depends upon the mean value of the Boss declinations.

Comparing (4) and (5) we see that there is a difference in P.M. given by

$$\text{Boss} - \text{FK}_3 (\text{P.M.}) = 14 \sin \theta - 6 \cos \theta - 19, \quad (6)$$

the unit being ".01 per century, so that much of the difference in declination for epoch 1950 is due to the difference in P.M. as we should expect.

It will be seen that (2) and (6) are not very different, except in the constant term. We would emphasize, however, that the constant term in (2) does not depend upon the Cookson observations, as it has not been assumed that the mean value of the latitude is constant over a long period of time.

PRELIMINARY VALUES OF THE VARIATION OF LATITUDE
AT GREENWICH DURING 1936-1938, TOGETHER WITH
AN ACCOUNT OF THE NEW OBSERVING PROGRAMME.

H. R. Hulme, M.A., Ph.D.

(Communicated by the Astronomer Royal)

1. Description of New Programme

In 1936 the Cookson floating telescope for the determination of the variation of latitude was moved from the courtyard of the Royal Observatory to a new site in the Christie Enclosure. At the same time a new programme of observations was introduced, as it was now no longer necessary to determine the constant of aberration.

The purpose of this note is to give a short account of the results obtained during the first two years of observations based on the new programme.

The stars chosen for observation are taken mainly from the new Boss General Catalogue, advance values for the Declinations and Proper Motions having been kindly supplied by Professor B. Boss. The programme contains 148 pairs in all, but of these, 36 pairs were observed on the previous Cookson programme for a number of years, and for these pairs the mean zenith distances and proper motions are taken from the latitude observations. These values are considerably more accurate than any which could be obtained by ordinary meridian observations of the separate stars. Nine pairs contain stars not in the Boss General Catalogue, and data for these have been obtained from other catalogues.

The star pairs are divided into 24 groups, each containing from 10 to 18 stars, extending over a period of about one hour in R.A. At present the arrangement is that three groups are observed each night. Starting, for example, with groups 6, 7, 8, after about a fortnight this is replaced by 7, 8, 9, and so on. Each group is therefore observed over a period of about six weeks.

This programme requires one observer for each night, the observations being centred round midnight, whereas the previous programme involved one evening and one morning observer. A corresponding disadvantage arises, however, because each group is observed for six consecutive weeks in each year. This results in the group corrections not being so well determined as when each group is observed in two periods separated by two to four months, as in the old programme. It should be added that in the old programme the groups were observed as near to 6 a.m. and 6 p.m. as possible, in order to obtain a large aberration factor. The systematic difference between evening and morning latitudes is not now regarded as being due to an error in the constant of aberration, so that a large spread in the times of observation of each group is no longer important from this point of view.

2. *Discussion of Results*

In dealing with the results obtained for the first two years, no attempt has been made to obtain the group corrections and latitude variation by the method of least squares, as used in treating the results of previous years (see the Greenwich publications on the Variation of Latitude). It is not possible to obtain, in this way, group corrections of any accuracy from two years' results only. Instead we have taken the 36 pairs observed also in the previous programme to form a basis for the latitude variation. The reductions have been made as follows:—

(i) It is first necessary to bring the "old" stars and the "new" stars to the same system of declinations. As stated in the previous paper, the system of the Boss General Catalogue requires a correction to declination which is periodic in R.A. and is given by

$$+ 12.8 \sin \theta + 5.4 \cos \theta,$$

where θ is the R.A. and the unit is $0''.01$. We therefore apply first this correction to the Boss places.

We then find for the 36 stars common to the old and new programmes

$$\left. \begin{array}{l} \text{Mean Z.D. north calculated from the cor-} \\ \text{rected Boss declinations for an assumed} \\ \text{value of the latitude} = 51^\circ 28' 38''.03 \end{array} \right\} - \left\{ \begin{array}{l} \text{Mean Z.D. north found from} \\ \text{observation on the old site} \end{array} \right.$$

$$= 1''.01 \pm 0''.02.$$

We therefore add $1''.01$ to the values taken from the Cookson observations to bring them to the same system as the (corrected) Boss declinations. The 36 pairs common to the two programmes therefore give the latitude of the old Cookson site as $51^\circ 28' 39''.04$, in agreement with the latitude derived in the preceding paper. The preliminary value for the latitude of the new site was obtained by measuring its distance south of the old site, which is 101 feet approximately. As the ground slopes away rather quickly to the north of the old site, we should expect the latitude of the new site to be rather less than $51^\circ 28' 38''.03$, the value assumed (see § 3).

(ii) Having obtained the ephemerides of all the star pairs, the residuals (observed *minus* tabulated mean zenith distance north for each pair) were first corrected to reduce the mean of each pair to the mean of the group for reasons given in the Greenwich publications quoted.

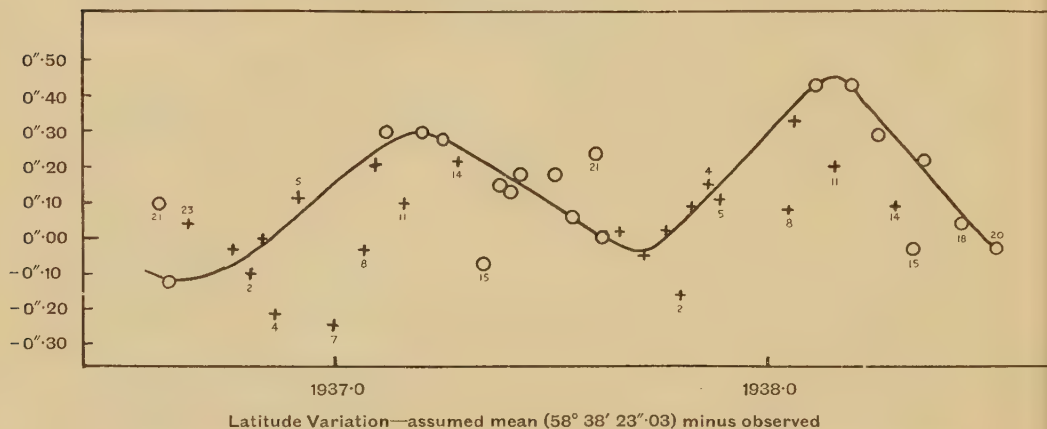
(iii) For those groups which contain two or more pairs from the old programme, a group correction was arrived at on the assumption that these pairs required no correction. As each group has been observed for two periods (one each year), this procedure is equivalent to making the mean latitude of the two periods equal to the latitude given by the "old" star pairs only, the difference between the two periods, however, depending on all the pairs observed.

(iv) For groups consisting entirely of new pairs, and two groups each containing only one "old" star, group corrections were adopted to bring the two values of the latitude as near as possible to the curve given by the

“old” pairs. In the five months September to January there are only three old pairs on the programme, so that in this region the latitude variation and group corrections are rather weak.

3. Results

The results are shown in the figure below, in which the assumed mean *minus* the observed latitudes are plotted.



The circles represent results obtained from the old Cookson stars as explained in § 2 (iii), the group corrections having been already applied. The crosses represent values for the groups containing new stars only, or not more than one old star, *before* any group corrections have been applied. To avoid overcrowding, only a selection of the groups have been numbered.

From the figure the (assumed) mean latitude, $51^{\circ} 28' 38''.03$, is seen to be too large by about $0''.10$, as anticipated in § 2 (i).

The preliminary group corrections are given below :

Group	Correction (Unit = $''\cdot 01$)	Group	Correction (Unit = $''\cdot 01$)
I	0	XIII	+ 19
II	+ 13	XIV	+ 10
III	0	XV	- 19 + 25
IV	...	XVI	- 15
V	0	XVII	- 2
VI	...	XVIII	- 2
VII	...	XIX	...
VIII	+ 26	XX	- 1
IX	0	XXI	+ 2 - 20
X	+ 8	XXII	+ 6
XI	+ 20	XXIII	- 10
XII	+ 25	XXIV	0

The figures in *italics* represent group corrections which have already been added to obtain the points marked with circles in the figure. For groups XV and XXI a further correction seems to be required, and for

groups IV, VI, VII and XIX the group corrections are too weak to be of any value.

The smoothed values of the corrections to be applied to the assumed mean latitude of $51^{\circ} 38' 28''.03$, or alternately to the observed declinations reduced with this value of the latitude, are as follows :—

1936.6	+ 0".12	1937.6	- 0".02
36.7	+ 0".11	37.7	+ 0".03
36.8	+ 0".05	37.8	- 0".04
36.9	- 0".05	37.9	- 0".17
37.0	- 0".16	38.0	- 0".29
37.1	- 0".25	38.1	- 0".40
37.2	- 0".30	38.2	- 0".44
37.3	- 0".25	38.3	- 0".26
37.4	- 0".18	38.4	- 0".12
37.5	- 0".10	38.5	+ 0".02

A further investigation was made in order to see if there was any "wind effect" in the new site. To do this the latitude variation was subtracted from the corrected mean residuals of each group for each night, and the resulting residuals correlated with the direction of the wind at the time of observation. A definite wind effect is apparent, the residuals (observed *minus* tabulated mean zenith distance north) being about $0''.07$ greater for N. winds than for S. winds. The data are not sufficient, however, to justify the adoption of any functional relationship between the mean residuals and the direction of the wind. The values of the latitude given above are not corrected for wind effect.

THE CONSTANT OF NUTATION.

Harold Jeffreys, F.R.S.

Some points in Dr. H. S. Jones's redetermination of this important constant* seem to merit further discussion, namely, the validity of the estimated uncertainty, and the method of combining two sets of determinations when part of the data are common to both. The uncertainty of a mean or other summary, found on the hypothesis that all the errors of observation are independent, is very liable to be too low. In the simplest case, that of two measures derived from a law involving a positive correlation r , the standard error of the mean of two errors x_1 and x_2 is given by

$$\begin{aligned}\sigma^2(\bar{x}) &= \frac{1}{4}\sigma^2(x_1) + \frac{1}{4}\sigma^2(x_2) + \frac{1}{2}r\sigma(x_1)\sigma(x_2) \\ &= \frac{1}{2}\sigma^2(1+r)\end{aligned}$$

if the standard errors are equal. For the mean of n observations, similarly,

$$\begin{aligned}\sigma^2(\bar{x}) &= E\left[\frac{1}{n^2}\sum x^2 + \frac{2}{n^2}\sum x_1x_2\right] \\ &= (\sigma^2/n)\{1 + (n-1)r\},\end{aligned}$$

E denoting the expectation, given the parameters in the law of error. In the extreme case $r = 1$ the uncertainty of the mean is equal to that of one observation. But even with small values of r the second term is liable to mount up when many observations are combined, and the estimate $\sigma/\sqrt{n}$ should not be trusted until some check has been applied that shows the correlation to be in fact negligible. In cases studied by Pearson the means of 25 observations were no steadier than those of 2 to 15 independent observations should have been.† This kind of internal correlation is neither an accidental nor a systematic error in the usual sense,‡ but it is clear that its effects on the actual uncertainty, as opposed to that found on the hypothesis of complete

* *M.N.*, 98, 440-447, 1938.

† K. Pearson, *Phil. Trans.*, A, 198, 235-299, 1902; Jeffreys, *Proc. Roy. Soc.*, A, 165, 187-190, 1938.

‡ An accidental error, as I understand it, is one such that, given a true value and a standard error, errors of all observations are independent. A systematic one is one that, if known exactly for one observation, would be determined for all others. An internal correlation may lead to a high positive correlation between neighbouring observations, but be negligible or reversed between those widely separated in the series, the changes occurring at irregular and unpredictable intervals.

This paper deals with the matter in terms entirely of internal correlation. In Dr. Jones's further note, which accompanies this, he connects the unexplained variation with an irregular variation of the diurnal variation of latitude. The reduction of the residuals at this stage indicates that, subject to the latter variation being known, the nutation residuals can now be treated as accidental and the apparent uncertainty is valid.

independence of the errors, may be very serious. Beyond values of n greater than about $1/r$ there will be very little gain in accuracy by increasing the number of observations. (For r negative there is an upper limit to the number of variables that can have correlation r , but there is no such limit for r positive.)

In this case I computed calculated values $C = A\bar{\gamma}$ for each of the weighted equations of condition in Dr. Jones's Table III, using the respective mean values of γ at the head of p. 445; if the equation is

$$A\gamma = B,$$

then $B - C$ is a random variation with standard error $\sigma\sqrt{A}$, where σ corresponds to unit weight, and σ can be estimated from the equation

$$\sigma^2 \Sigma A = \Sigma (B - C)^2.$$

Applying this to the three sets of data I get

$$\sigma = 0''.090, \quad \sigma = 0''.154, \quad \sigma = 0''.116.$$

These are slightly too low because the uncertainty of the means is excluded, but we can allow for this by multiplying by $(8/7)^{\frac{1}{2}}$, giving

$$\sigma = 0''.096, \quad \sigma = 0''.165, \quad \sigma = 0''.124.$$

Jones gives the probable error corresponding to unit weight as $0''.052$, corresponding to a standard error of $0''.077$. Thus all the estimates from the equations of condition are substantially higher. A convenient way of testing the significance of the difference is given by Fisher's z test; * if the observations are genuinely independent, a difference of the standard errors is a random variation. Fisher takes the number

$$z = \log_e \sigma_1 / \sigma_2,$$

where σ_1 is the larger of the estimates. If σ_1 is determined from n_1 degrees of freedom (here the number of equations of condition reduced by 1, that is 7) and σ_2 from n_2 (here unstated but very large), we have, for the three series respectively, $z = 0.223, 0.761, 0.476$. The 5 per cent., 1 per cent. and 0.1 per cent. points for z on the hypothesis of random variation, with these numbers of degrees of freedom, are at 0.38, 0.54 and 0.70. The first value found would therefore call for no special comment, but the third is nearly up to the 1 per cent. limit and the second beyond the 0.1 per cent. one. It therefore seems that some internal correlation has increased the fluctuations of the means. On the other hand, the results appear to be consistent among themselves. The first and second series give $z = 0.538$; the 5 per cent. point for the comparison of two series from 7 degrees of freedom each is at 0.667. The data may be combined without allowance for a difference of the standard errors for unit weight, but the uncertainty must be determined from the equations of condition and not from the original residuals.

The observations used overlap, nearly all the data of the first series and more than half those of the second being included in the third. Jones allows

* *Statistical Methods for Research Workers*, p. 230 and Table VI, 1936.

for this, but the method is not altogether satisfactory. Finding that 0.813 of the data represented by his first equation for the first series are included in the third, he multiplies the first equation by 0.187 and then adds it to the third. But the common part of the two series will not have the same mean as the whole of either, and it is the difference between the other parts that produces the difference in the means of the whole series. The method used gives weight 0.187 to an observation in the other part of the first series, and weight 1 to one in the other part of the third, whereas they should have equal weight. The tendency of the method will therefore be to overweight the third series.

I think the proper procedure is to introduce the mean of the common part as an additional unknown and try to estimate it. Without attention to the special complications of the present problem, suppose that we have two series of n_1 and n_2 observations of unit weight, with a common part n_{12} . We want to estimate a mean x . Then the equations as they stand can be written

$$\sqrt{n_1}x = \sqrt{n_1}x_1, \quad \sqrt{n_2}x = \sqrt{n_2}x_2, \quad (1)$$

each with unit weight, where x_1 and x_2 are the means of the respective series. But actually these are not independent, and there is a further unknown, the mean of the n_{12} common observations; call this x_{12} , and call the means of the $n_1 - n_{12} = n_1'$ and $n_2 - n_{12} = n_2'$ independent observations x_1' and x_2' . Then we have

$$n_1x_1 = n_1'x_1' + n_{12}x_{12}, \quad n_2x_2 = n_2'x_2' + n_{12}x_{12}, \quad (2)$$

and three equations of condition

$$\sqrt{n_1'}x = \sqrt{n_1'}x_1', \quad \sqrt{n_2'}x = \sqrt{n_2'}x_2', \quad \sqrt{n_{12}}x = \sqrt{n_{12}}x_{12}. \quad (3)$$

In the first two of these we eliminate x_1' and x_2' by means of (2); then we have

$$\sqrt{n_1'}x + \frac{n_{12}}{\sqrt{n_1'}}x_{12} = \frac{n_1x_1}{\sqrt{n_1}}, \quad (4)$$

$$\sqrt{n_2'}x + \frac{n_{12}}{\sqrt{n_2'}}x_{12} = \frac{n_2x_2}{\sqrt{n_2}}, \quad (5)$$

$$\sqrt{n_{12}}x - \sqrt{n_{12}}x_{12} = 0, \quad (6)$$

each with unit weight, to find x and x_{12} . Forming the normal equations we have

$$(n_1' + n_2' + n_{12})x + n_{12}x_{12} = n_1x_1 + n_2x_2, \quad (7)$$

$$n_{12}x + n_{12}^2\left(\frac{1}{n_1'} + \frac{1}{n_2'} + \frac{1}{n_{12}}\right)x_{12} = \left(\frac{n_1x_1}{n_1'} + \frac{n_2x_2}{n_2'}\right)n_{12}x_{12}, \quad (8)$$

giving

$$x = \frac{(n_1 - n_{12})x_1 + (n_2 - n_{12})x_2}{n_1 + n_2 - 2n_{12}}; \quad \text{weight} = \frac{n_1 + n_2 - 2n_{12}}{1 - n_{12}^2/n_1n_2}. \quad (9)$$

To estimate n_{12} in the present case, about $\frac{1}{3}$ of the weight for the first

period in the first line of Jones’s table reappears in the third period ; hence we take in this line

$$n_{12} = \frac{11}{13} \times 24.77 = 20.96, \qquad n_3 = 41.52,$$

and proceed analogously in other cases.

The results of the combination are then γ_{12} , as follows :—

γ_{12}	Weight	γ_{123}	Weight
−.0158	43.1	−.0202	44.5
+.0222	18.4	+.0352	24.8
+.0469	21.0	+.0417	22.6
+.0103	21.2	+.0011	24.7
−.0016	6.2	−.0173	8.4
+.0353	4.4	+.0489	5.3
+.0410	3.0	+.0428	3.1
−.0097	17.8	−.0082	18.2

The weighted mean is +0".0076, weight 135.1. The second series is then combined with the result, giving the results γ_{123} . The weighted mean is now +0".0069, weight 151.6. It seems peculiar that the second series should make so little difference to the mean, seeing that its own mean is +0".0028 ; but those groups that contain most information not included in the third series differ much less in the mean. The standard error for unit weight is found from the residuals to be

$$\sigma = 0".135(8/7)^{\frac{1}{2}} = 0".144.$$

Compared with the apparent uncertainty, it gives $z = 0.63$, which is beyond the 1 per cent. limit. There seems to be adequate reason for asserting the existence of an internal correlation.

My own tests, at small numbers of groups, tend to assert significance at rather lower values of z than the 5 per cent. limit, though not with much confidence.* There is not much difference in practice, but Fisher’s has the advantage that tables exist for it ; but my own would certainly lead to the same conclusion.

The resulting value of the nutation constant is +9".2169 ± 0".0117 (standard error) or ± 0".0079 (probable error). The uncertainty given by Jones is nearly doubled, but there is little change in the mean. The corresponding value of μ^{-1} is 81.53 ± 0.33 or 81.59 ± 0.33 (standard errors) according as we adopt Newcomb’s or Oort’s value of the constant of precession. It should be noticed that when a standard error is estimated from a small number of degrees of freedom, as in this case, the posterior probability of the true value is distributed according to “Student’s” law with 7 degrees of freedom instead of the normal law, on account of the uncertainty of the standard error.† To establish a probability of 0.95 that the true value is

* *Proc. Roy. Soc., A*, **167**, 379, 1938. For the t rule, of which the z rule is an extension, the 5 per cent. limit for 8 observations corresponds to about 2 to 1 odds on the need for a new parameter.

† Jeffreys, *Scientific Inference*, p. 69 ; *Proc. Roy. Soc., A*, **160**, 325–348, 1937.

within a given range, we must go 1.96 times the standard error on each side of the most probable value, if the standard error is accurately known; but when it is found from only 7 degrees of freedom we must replace this by 2.365.*

I express no opinion on the explanation of the internal correlation; errors in the proper motions and other possible systematic errors appear to have been adequately allowed for in the estimated uncertainty. Its existence, however, seems to be beyond reasonable doubt. The apparent uncertainty of an estimate found by combining many observations should, however, be mistrusted until the χ^2 or z tests or my modifications have been applied to test whether there is some such correlation; the postulate of independence of the errors is always under suspicion until some sensitive check on it has been applied.

At the same time I should like to mention the desirability of abandoning the use of "probable errors." At best the multiplication of the standard error by 0.6745 is a useless complication and is valid only when the standard error is accurately known. When a mean is estimated from two observations, the accuracy being otherwise unknown, the probability, given these observations, that the true value lies between them is $\frac{1}{2}$; in this case the probable error is equal to the standard error. The standard error has a definite meaning in terms of the observations, and all the recognized statistical tables are stated in terms of it. In applying the χ^2 or any similar test the factor applied must be first removed.

Since the weights of the values of γ are very different, it may appear that the effective number of degrees of freedom used in determining the standard error is less than 7. On the other hand, we should be entitled to suppose that the standard error of a mean is $\sigma k^{-\frac{1}{2}}\{1 + (k-1)r\}^{\frac{1}{2}}$, where σ is the standard error on the hypothesis of independence, already known here to be 0".077, and k the effective number of observations in the group. Then it would be possible to write down the equations to determine r , which would replace the estimation of a standard error, and as far as γ is concerned the effect would be to reduce the weight of the numerous groups and increase that of the smaller groups. It appears that here r is about +0.07; on this basis the relative weights vary from about 4 to 12, and the solution is $\gamma = +0".0109 \pm 0".0103$. The estimate of γ is somewhat increased and its standard error diminished. The complete form of the posterior probability for γ , however, cannot be found in finite terms, and I have not pursued the matter further.

* Fisher, *loc. cit.*, Table IV.

THE DETERMINATION OF THE CONSTANT OF NUTATION FROM THE GREENWICH LATITUDE VARIATIONS (SECOND PAPER).

H. Spencer Jones, F.R.S., Astronomer Royal.

A determination of the constant of nutation from the latitude variation observations obtained with the Cookson floating telescope at Greenwich in the period 1911-1936 was given in *M.N.*, **98**, 440, 1938. It may be recalled that the derivation of the nutation was based on the differences between the latitude variations derived from evening and morning observations at the same epoch in successive years; the difference between the nutation factors of the evening and morning pairs varies in an approximately harmonic manner with the period of the Moon's node, which makes it possible to separate the correction to the nutation factor from the correction to the positions at mean epoch and the correction to the proper motions.

The discussion was restricted to star-pairs that were observed for a number of years, so that no variable element was introduced by change of star-pairs. By differencing the evening and morning observations, errors in the adopted latitude variations were almost completely eliminated; they could only come in through the differential corrections applied to reduce the mean epoch of the morning observations to the mean epoch of the evening observations, the differences of these epochs being usually less than 0.03 year. Corrections were applied for the wind effect and also for the diurnal variation of latitude.

It seemed therefore, and it was accordingly assumed, that the determination of the constant of nutation in this way was entirely free from errors of a systematic nature. The constant of nutation is, in fact, generally regarded as a quantity which can be determined free from systematic error, because its derivation is based on observations made from year to year at the same time of day and at the same season of the year; the constant of aberration, on the other hand, is peculiarly difficult to determine free from systematic error, because its derivation is based on observations made at different times of the day and at different seasons of the year.

There were considerable changes of the star programme for the Greenwich latitude variation observations in 1919 and 1929. The derivation of the nutation constant was therefore based on (a) all star-pairs observed during the interval 1911-1928; (b) all star-pairs observed during the interval 1920-1936; (c) all star-pairs observed for the whole period of the observations 1911-1936.

Eight combinations of groups were used, and the probable errors corresponding to unit weight inferred from each of these combinations were in close agreement and in accordance with the value to be inferred from the probable error of a single latitude plate. This was accepted as satisfactory,

though the individual values of the nutation corrections derived from the various combinations differed rather considerably.

Dr. Jeffreys * has pointed out that a test of the significance of the differences between the individual values of the nutation correction, derived from the several combinations of groups by means of Fisher's z -test, indicates that there is some internal correlation; this would imply that the results from the different combinations of groups are not genuinely independent.

It may here be remarked that though the different combinations of groups gave rather widely varying corrections to the nutation, the sign of the correction for a given combination of groups tended to be the same whether the correction was derived from the interval 1911-1928, or 1920-1936, or 1911-1936 (see *M.N.*, 98, 444, Table III).

The only possible source of error that can be left in the observational material which could produce an effect of a systematic nature on the constant of nutation by the method used for its derivation is a slow irregular fluctuation in the diurnal variation of latitude. In a given combination of groups the observations are made at about the same epochs each year and at about the same time of the night. A correction was applied in the discussion for the mean diurnal variation of latitude. But if the difference between morning and evening latitudes at the same season of the year fluctuates from year to year in a manner that is not accidental, a correlation with the difference of evening and morning nutation factors is possible, leading to a spurious value of the nutation constant.

The nutation factor has the form $A \cos \alpha + B \sin \alpha$, where A and B involve the longitude of the Moon's node and the longitude of the Sun. A regular fluctuation in the diurnal variation of latitude would therefore lead to corrections of the same amount but of opposite sign for groups whose right ascensions differed by 12 hours. In general, the effect on the nutation constant might be represented as a first approximation by the expression $a \cos \alpha + b \sin \alpha$, where a and b are quantities not involving the right ascension, which will vary from one period of observation to another.

To obtain some evidence as to the possible existence of such a fluctuation in the evening and morning differences, the mean differences for each combination of groups, as used for the determination of the correction to the nutation, were tabulated together for each year. For the first few years there was a predominance of negative differences, whilst around 1920 to 1922 there was a considerable predominance of positive differences. For the three periods, 1911-1919, 1919-1927, 1927-1936, into which the observations were divided in the detailed discussions there were respectively 29 positive, 37 negative differences; 37 positive, 27 negative; 35 positive, 32 negative.

In the detailed discussion of the zenith telescope observations for the derivation of the latitude variation a term was included of the form $S \sin (\odot - \alpha)$ in the analysis of the tabular *minus* observed zenith distances, to take account of diurnal variation of latitude. The values of S derived from the discussion were as follows:—

* See the preceding paper.

$$\begin{aligned}
 1911.7-1919.4 & -\cdot 0085 \pm \cdot 0047 \\
 1919.4-1927.0 & -\cdot 0222 \pm \cdot 0045 \\
 1927.0-1936.5 & -\cdot 0149 \pm \cdot 0038
 \end{aligned}$$

The differences between the values of S were not regarded as significant in relation to their probable errors. Now, since $\sin (\odot - \alpha)$ is negative for the evening observations and positive for the morning observations, the effect of the differences in S on the evening *minus* morning latitude differences used for the derivation of the correction to the nutation will be that the latitude differences will tend to be largest (algebraically) for the period 1919.4-1927.0, and smallest for the period 1911.7-1919.4. This is exactly what the evening *minus* morning differences show.

The conclusion is that the differences between the three values of S , the coefficient of the diurnal latitude variation term, may be real and that they afford some support for the assertion that the diurnal variations are subject to fluctuations that are not purely accidental in character. Some support for this conclusion is provided by the closing errors in the observations of the International Latitude Service. The closing error in these reductions is the sum of the differences of the mean morning and evening latitudes for the twelve periods of observation through a year; the group being observed in cyclical rotation, the closing error is independent of group corrections and of latitude variation, except diurnal variation. The mean closing errors for each station, obtained by starting with each group in turn and summing to the corresponding group in the next year and taking the mean of the twelve closing errors so derived, for the period 1915-1922 are as follows :— *

International Closing Errors

Year	Mizusawa	Tschardjui	Carloforte	Gaithersburg	Cincinnati	Ukiah
	"	"	"	"	"	"
1905-06	-.209	-.529	-.351	-.005	-.169	-.352
06-07	-.027	-.271	-.255	-.107	-.385	-.240
07-08	-.091	-.132	-.224	-.053	-.403	-.264
08-09	-.323	-.391	-.117	+.018	-.265	-.364
09-10	-.046	-.454	-.167	-.089	-.349	-.546
10-11	-.263	-.493	-.209	-.120	-.203	-.359
11-12	-.387	+.014	-.137	-.260	-.019	-.377
12-13	-.275	-.265	-.189	-.256	-.160	-.179
13-14	-.286	-.689	-.214	-.333	-.053	-.295
14-15	-.257	-.989	-.284		-.110	-.544
15-16	-.341	-.532	-.303			-.573
16-17	-.356	-.425	-.210			-.434
17-18	-.373	+.011	-.348			-.733
18-19	-.359		-.361			-.324
19-20	+.025		-.211			-.291
20-21	-.271		-.109			-.143
21-22	-.159		-.189			-.225

* These values are derived from the data in *Ergebnisse des Internationalen Breiten- dienstes*, Bd. V, pp. 74, 75, and Bd. VI, pp. 50-52 (Potsdam, 1916 and 1932).

Apart from accidental errors, the closing errors represent the sum of the mean evening and morning differences of latitude for the twelve periods of observation in a year. Such differences have been interpreted, in the discussion of the Cookson observations, as arising from a diurnal variation of the latitude. It will be noticed that for each of the stations the closing error fluctuates considerably from year to year and these fluctuations are analogous to the variations in the evening *minus* morning differences revealed in the Cookson observations.

If the observations could extend over several nutation periods the effects of the fluctuation in the diurnal term could be neglected because they would not be likely to correlate over a long interval with the differences of the evening and morning nutation factors. But with observations limited to twenty-five years a correlation is possible. The observational material cannot distinguish between the portion of the differences that is due to the adopted value of the nutation constant being in error and the portion due to latitude changes between evening and morning observations, unless some law for the effect can be assumed.

We have seen above that, in so far as there is correlation between the nutation term and changes in the diurnal variation, it can be assumed to a first approximation that the effect on the nutation is of the form $a \cos \alpha + b \sin \alpha$. We assume therefore that, for any given combination of groups, we do not derive γ , a correction to the constant of nutation, but a quantity $\gamma + a \cos \alpha + b \sin \alpha$.

An expression of this form was substituted for γ for each group, and the results given in Table III (*M.N.*, 98, 444, 1938) were analysed to determine γ , a and b . The results were as follows, where the a and b terms have been combined into a sine term :—

1911-1928	— " " $-.0050 + .0276 \sin (\alpha + 49^\circ.5)$
1920-1936	— $-.0133 + .0277 \sin (\alpha + 80^\circ.4)$
1911-1936	+ $.0044 + .0205 \sin (\alpha + 62^\circ.5)$
Combined <i>A</i>	+ $.0003 + .0220 \sin (\alpha + 68^\circ.4)$
„ <i>B</i>	— $.0008 + .0229 \sin (\alpha + 71^\circ.4)$

The combined *A* solution was obtained by combining the three partial solutions—the first two of which utilise some material that is also contained in the third—as explained in the previous paper ; the combined *B* solution was obtained from a rather more rigorous method suggested by Jeffreys.

Further solutions were made by neglecting the proper-motion corrections, with the following results :—

1911-1928	— " " $-.0009 + .0162 \sin (\alpha + 56^\circ.3)$
1920-1936	— $.0061 + .0252 \sin (\alpha + 50^\circ.5)$
1911-1936	+ $.0054 + .0209 \sin (\alpha + 57^\circ.3)$
Combined <i>A</i>	+ $.0016 + .0221 \sin (\alpha + 57^\circ.0)$
„ <i>B</i>	— $.0024 + .0233 \sin (\alpha + 54^\circ.1)$

The mean probable error corresponding to unit weight obtained from

the various solutions is now $\pm 0''.059$. The internal probable error is not affected and remains, as before, $\pm 0''.052$. The difference between these two values is not significant and suggests that errors of a systematic nature have been effectively eliminated by the inclusion of the term depending on the right ascension.

The adopted proper motions, based on the preliminary discussion of twenty-five years' observations, are likely to be more accurate than the values derived from the shorter periods, where partial correlation between the proper-motion term and the nutation term is possible. For the full period the results obtained by including proper-motion corrections are to be preferred.

The most satisfactory method of using the combined material is to combine the solutions for the periods 1911-1928 and 1920-1936, ignoring proper-motion corrections but with the reduced weights that are obtained when such a correction is inserted, with the solution for the whole period, 1911-1936, in which proper-motion corrections are included. The results then obtained by the two methods of combining the material are then as follows:—

$$\begin{aligned} A &+ 0''.0035 + 0''.0205 \sin(\alpha + 62^\circ.1), \\ B &+ 0''.0034 + 0''.0200 \sin(\alpha + 59^\circ.6). \end{aligned}$$

The probable errors corresponding to unit weight given by these two solutions are $\pm 0''.050$ and $\pm 0''.049$ respectively, in almost exact agreement with the probable errors inferred from the internal accordance of each series.

We therefore adopt as the most probable value of the correction to the nutation constant to be inferred from the Greenwich latitude variation observations the value

$$+ 0''.0034 \pm 0''.0042,$$

where $\pm 0''.052$ has been adopted as the probable error for unit weight.

The constant of nutation, the assumed value being $9''.210$, becomes

$$9''.2134 \pm 0''.0042.$$

The value previously obtained was $9''.2173$.

The revised value for the nutation gives a value for the reciprocal of the mass of the Moon of 81.64 on the basis of Newcomb's value for the luni-solar precession; or of 81.70 on the basis of Oort's value. These values have a probable error, in so far as the uncertainty in the nutation is concerned, of ± 0.12 .

In the derivation of the constant of nutation from the observations of the International Latitude Service it is possible to eliminate the variations of latitude due to the polar motion because the observations at the various stations can be combined. A correction to the adopted constant of nutation can be derived from the observations of each star-pair at each station. The discussion by Przybyłok* of the observations at five stations during the years 1900-1915 gave corrections to the constant of nutation which, for each

* *Zentralbureau der Internationalen Erdmessung*, Veröff. (neue Folge, Nr. 36), 1920.

of the five stations, showed a strong dependence on right ascension. It was assumed that the corrections could be represented by a constant term and a harmonic term in right ascension, the constant term being adopted as the true correction to the nutation. The harmonic term was attributed to the existence in the z -term of the latitude variation of components of periods of 0.73 and 1.13 years, in addition to the annual period. In the derivation of the constant of nutation by this method the x and y components of the latitude variation are eliminated, but the z -term is not eliminated. The observations of each group are made at about the same mean epoch each year, so that any local variation of latitude of a strictly annual period and constant amplitude would not affect the derived value of the nutation. But it is now known that the z -term is only represented to a first approximation by an annual period, and that it can vary considerably in its amplitude and in the nature of its variation from year to year. Such fluctuations in the local variation of latitude must affect the derivation of the nutation constant and, just as in the Cookson observations, the effect on the derived constant will be approximately a harmonic variation in right ascension. It is not necessary to suppose, as Przybyllok did, that there are regular terms with periods differing from a year in the local variations.

Local variations of this nature are eliminated automatically in the discussion of the Greenwich observations; on the other hand, the fluctuations in the evening and morning differences which affect the Greenwich results do not enter into the International results, because the observations at the international stations are always made near midnight.

Summary.—In observations for the variation of latitude, the latitude variation and nutation can be separated by making observations in different longitudes at the same right ascension (as in the International observations) or by making observations at different right ascensions at one station (as at Greenwich). For the determination of the nutation, observations should be extended over at least nineteen years. By the first method, fluctuations in the local variation of latitude (z -term) at any station, and by the second method, fluctuations in the evening and morning differences (diurnal variation of latitude), will tend to affect the derived constant of nutation in a systematic manner. In each case the derived values will show a variation with right ascension. Effects of this nature are found to exist both in the Greenwich and International Latitude results. When allowance is made for this effect, the constant of nutation inferred from the Greenwich results becomes $9''.2134 \pm 0''.0042$.

NOTE ON THE ZEEMAN EFFECT IN SUNSPOT SPECTRA.

J. Evershed, F.R.S.

Photographs of sunspot spectra obtained with the liquid prism spectrograph have clearly shown the Zeeman widening of several lines in the region near $H\beta$ and $H\gamma$, where a dispersion of 1.5 to 2 mm./Å. has been obtained with four transmissions through the prism.

Recently it has been possible to obtain very high dispersion in the less refrangible regions after perfecting the optical performance of the prism. In the region between λ 5200 and 5300 the spot spectrum has been photographed many times with ten transmissions through the liquid prism, the scale here being 2 mm./Å.

With this dispersion a very striking example of a Zeeman doublet is revealed in the *Fe* line λ 5250.218. This line, in all large spots examined, is split into two at the edges of the penumbra, and the separation increases to about 0.25 Å. over the umbra when the spot is not very far from the centre of the disc. This amount of separation of the components is much greater than in any other Zeeman doublet that I have observed. In some cases a central component appears on the umbra, even when the spot is near the centre of the disc; but there is some doubt whether this is due to superposed light from the photosphere scattered in a hazy sky and showing the unaffected line.

The neighbouring lines are also greatly widened and show signs of doubling in some spot spectra. These are 5247.060 due to iron and 5247.576 due to chromium. There is also an iron line at 5250.576, but this is only very slightly widened.

I cannot find any reference to the line 5250.218 in Hale's work on the magnetic field in sunspots, although it appears to be one of the most conspicuous cases of the magnetic effect to be found in the spot spectrum, and it would apparently serve better than any of the lines that had been chosen in attempts to measure the general magnetic field in the Sun. The only references to it that I can find are in a list of lines affected in sunspots by W. M. Mitchell.* Also in a spot observed on 1909 May 11 the line is described by Mitchell as a wide doublet in one position in the spot group and a wide triplet in another. The line is unaccountably omitted in Adams's comprehensive list of lines doubled in spots.†

As regards the Zeeman effect observed in the laboratory, King has measured the magnetic separation of this line, and some of the neighbouring iron lines, in furnace spectra.‡ His results are as follows, observed along the lines of force, and with a field strength of 6400 gauss:—

* *Ap. J.*, 24, 86, 1906. † *Ap. J.*, 30, 120, 1909. ‡ *Ap. J.*, 51, 115, 1920.

λ	$\Delta\lambda$
5232.954	0.20
5247.060	0.34
5250.218	0.51
5254.955	0.40
5270.390	0.12

Here also the line 5250.2 shows a wider separation of the two components than any of the nearby lines, and the relative separations in these five iron lines agree well with those shown in my spot spectra. The line 5250.656 was not measured, presumably because so little affected.

In Hale's work on the general magnetic field of the Sun it is curious to note that the chromium line at 5247.576 and the iron line at 5250.656 were both measured, whilst 5250.218 was not measured although so conspicuously affected in spots. This may possibly be due to the fact that the magnetic effect of the line had not then been observed in the laboratory: King's measures of it in the furnace spectrum were made much later. The separation of the components is notably greater than that of the chromium line in my spectra, and at least three times greater than that of its near neighbour 5250.656. It is a relatively narrow line in the Sun's spectrum, and should be easy to measure with precision. The region of spectrum including the line occurs on some of the original negatives of the solar magnetic field taken at Mt. Wilson in the year 1913, which Dr. Hale left with me in 1933. I hope to undertake some new measures of these plates and compare the shifts of 5250.2 with its near neighbours.

Summary.—The liquid prism spectrograph has revealed the Zeeman effect in many lines in the $H\beta$ and $H\gamma$ regions of the sunspot spectrum with four transmissions through the prism.

In the less refrangible regions and with ten transmissions a dispersion of 2 mm./Å. reveals a striking example of a Zeeman doublet in the iron line at 5250.218. This line does not appear to have been observed by Hale in his work on the magnetic field in sunspots, nor was it included in the lines used by him in investigating the general magnetic field of the Sun, although its close neighbours showing far less effect in spots were used and gave evidence of the general field.

The Zeeman effect of the line in the furnace was shown by King to give a wider separation than any of the nearby lines.

It is suggested that this line would be very suitable for obtaining new evidence of the general magnetic field of the Sun.

THE OXFORD SOLAR TELESCOPE AND HARTMANN TESTS OF ITS PERFORMANCE.

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Introduction.—The use of a fixed telescope, fed either by a siderostat or a cœlostat and auxiliary plane mirror, is of long standing in solar and eclipse spectroscopy. It was not, however, until the erection of the Snow Telescope at Mt. Wilson that a critical study of the definition given by such instruments was made. Hale* then showed that there were two characteristic defects of this instrument, the first and more serious being a change of figure of the two flats with the heat of the Sun, leading to changes of focus and astigmatism, and the second arising from local convection currents, much accentuated in the vicinity of the ground. To avoid this second defect he built the 60- and the 150-foot Tower Telescopes; so successful have these two instruments proved that similar telescopes have been erected at Potsdam, Arcetri and Canberra.

It must be admitted, however, that the solar definition given by the tower telescope, if direct photographs are any criterion, often falls short of that given by the ordinary equatorial refractor. There is therefore probably need for further experiments in the design of such instruments, and some interesting experiments have recently been reported by Hale† and McMath,‡ but especially does there appear to be need for optical tests of the whole optical train of the solar telescope as a unit, in order to have a comparison with the similar tests of the ordinary refractors and reflectors. It is the purpose of the present paper briefly to describe a solar telescope with one or two slight modifications of design which has recently been erected at Oxford, and to give the results of Hartmann tests of this instrument.

Section 1.—Design and Installation of Telescope

A solar telescope, to be used as an auxiliary to a large fixed spectroscope, must satisfy four main requirements. First, it should be of the tower type; secondly, it should give a linear scale of the order of 1 mm. = 10" of arc; thirdly, it should be strictly achromatic; and finally, the optical system should suffer the minimum distortion under the heat of the Sun. The design of the present instrument has been determined by these four conditions, subject to the overriding requirement that the total cost of the telescope and its installation should not exceed £1800.

* G. E. Hale, *Ap. J.*, **21**, 124, 1905; **23**, 6, 1906; **25**, 68, 1907; **27**, 204, 1908. *Mt. Wilson Contr.*, Nos. 1, 4, 14 and 23.

† G. E. Hale, *Ap. J.*, **82**, 127, 1935.

‡ R. R. McMath, *Pub. Obs. Un. Mich.*, **7**, No. 1, 1937.

(i) *The Tower*.—Since a double steel tower of the Mt. Wilson pattern would have absorbed too large a proportion of the money available, it was decided to build the telescope into the existing east tower * of the observatory. This housed under its original dome the De la Rue Speculum Metal Reflector, and beneath this an office and two dark rooms; it was evident that if this instrument were taken down and the floors removed there would be available a substantially built tower, isolated from the rest of the building, with a central brick pier suitable for carrying the solar telescope. The work of clearing and making the necessary minor alterations was entrusted to a local builder in 1934, the De la Rue telescope having previously been taken down and presented to the Museum of the History of Science. The tower thus made available is 18.6 feet (5.7 m.) square internally by 24.5 feet (7.5 m.) high. The original papier-mâché-covered dome, with a shutter sliding in three sections, revolves on top of the 18-inch-thick tower walls, and may be turned from the floor of the tower by an endless rope. The original brick pier, 6 feet (1.8 m.) square at its base, tapers over its 21 feet (6.4 m.) height to 4 feet (1.2 m.) square at the top, where it is capped by a newly constructed concrete block to which the steel platform carrying the solar telescope is grouted. Entrance to the tower may be effected from the main building by a door at the basement level in the west wall of the tower, and from outside by a door in the east wall; the remaining doors and all the windows are bricked up flush with the walls.

(ii) *Scale* and (iii) *Achromatism*.—If the instrument is to be usable for the study of solar granulation, dimensions two to five seconds of arc, it is evident that the linear scale may be not smaller than one second of arc equals a tenth of a mm. This corresponds to a focal length of 20.6 m. (67.6 feet). Equally it is clear, since for spectroscopic work critical definition at all wave-lengths is essential, that the absence of achromatism in the visual objectives ordinarily used with tower telescopes cannot be tolerated; thus the visually focussed image given by such an objective of 30 cm. (12 inches) aperture and 18.3 m. (60 feet) focal length yields geometrical discs of confusion at λ 4000 of 1.8 mm. diameter, corresponding to 20" of arc. Quite apart from the study of minute detail, lack of achromatism may introduce serious errors into spectrophotometric studies.†

The requirement of a focal length of some 20 m. in a tower 7 m. high, coupled with complete achromatism, can only be met by the use of a Cassegrain reflecting telescope. The annular extra focal patch given by such an instrument on the collimator of the spectroscope is not a source of difficulty, but if the Cassegrain mirror is made large enough to give full illumination over a field half a degree in diameter there may be considerable light loss. This difficulty is avoided in Hale's "skew-Cassegrain" with its off-axis paraboloid,‡ but the cost and the difficulty of figuring such mirrors, so far

* See *M.N.*, 34, 49, 1873, for a description by C. Pritchard of the observatory building, and *M.N.*, 36, 1, 1875, for a photograph showing on the right the tower in question.

† E. Pettit and S. B. Nicholson, *Ap. J.*, 71, 153, 1930; *Mt. Wilson Contr.*, No. 397.

‡ G. E. Hale, *Ap. J.*, 82, 127, 1935.

only successfully carried out at Pasadena, rule this solution out of consideration in the present instance. Since it was proposed to use this telescope primarily for spectroscopic work, and since instantaneous direct photographs of the whole disc were not really essential, it was decided to make the Cassegrain mirror large enough to give, centred on the optical axis, a fully illuminated patch 5 cm. in diameter. The solar image thus obtained would show vignetting towards the edge of the field, but any part of the image brought on to the optical axis would be fully illuminated.

(iv) *Mirror Distortion*.—From trials with various glasses used as cœlostats and secondary mirrors in the Mt. Wilson solar telescopes, Pettit * has found changes of focus on continued exposure to the Sun ranging from 60 cm. for pyrex to 500 cm. for thin crown glass, in a total focal length of 45.8 m. (150 feet). It may then be shown that the corresponding radius of curvature of each of the flats must have been approximately 1490 m. for thin crown, 3170 m. for thick crown, and 13,800 m. for pyrex; if these mirrors are used in this condition at angles of incidence of 30° , about normal summer working conditions, the resulting astigmatism amounts to 16λ , 7.5λ and 1.6λ respectively. Recalling that the Rayleigh limit for critical definition is $\lambda/4$, one cause for the poor definition given by tower telescopes, even those with pyrex flats, is revealed.

Since the coefficients of expansion of crown glass, pyrex and fused silica are in the ratio of 21 to 7 to 1, it follows, as Hale pointed out over thirty years ago, and as is borne out by Pettit's observations, that mirrors of fused silica would show negligible changes of focus and astigmatism. In spite of the long-continued and costly experiments of the General Electric Co. at Lynn, Massachusetts, the difficulty has heretofore been to get satisfactory discs of this material. However, in 1933 the Thermal Syndicate Limited of Wallsend succeeded in making fused silica discs up to 53 cm. (21 inches) in diameter, and Mr. Cyril Young of Grubb, Parsons having drawn attention to this fact, it proved possible to use these discs in all five mirrors of the telescope.

Details of Design.—The optical parts, the detail of design and the construction of the mechanical parts of a solar telescope, conforming to the above general specifications, were entrusted to Sir Howard Grubb, Parsons & Co. of Newcastle. The contract was signed in the summer of 1933, and the instrument was completed and erected by them in the spring of 1935. The details of the resulting telescope will be made clear by reference to fig. 1, a diagrammatic sketch of the telescope as mounted in the east tower of the observatory, and figs. 2 and 3 (Plate 5), photographs of the steel platform and the main mirror cage.

The Cassegrain telescope proper consists of a paraboloid, A (figs. 1 and 3), of 31.8 cm. ($12\frac{1}{2}$ inches) clear aperture and 9.1 m. (30 feet) focal length. The fused silica disc is 4.4 cm. ($1\frac{3}{4}$ inches) thick. Vertically above it is the Cassegrain hyperboloid, B, of 12.7 cm. (5 inches) clear aperture, giving an equivalent focal length for the combination of 22.31 m. (73.1 feet), and an aperture ratio of $f/70$. The converging beam returning from

* E. Pettit, *Ap. J.*, 58, 208, 1923; *Mt. Wilson Contr.*, No. 266.

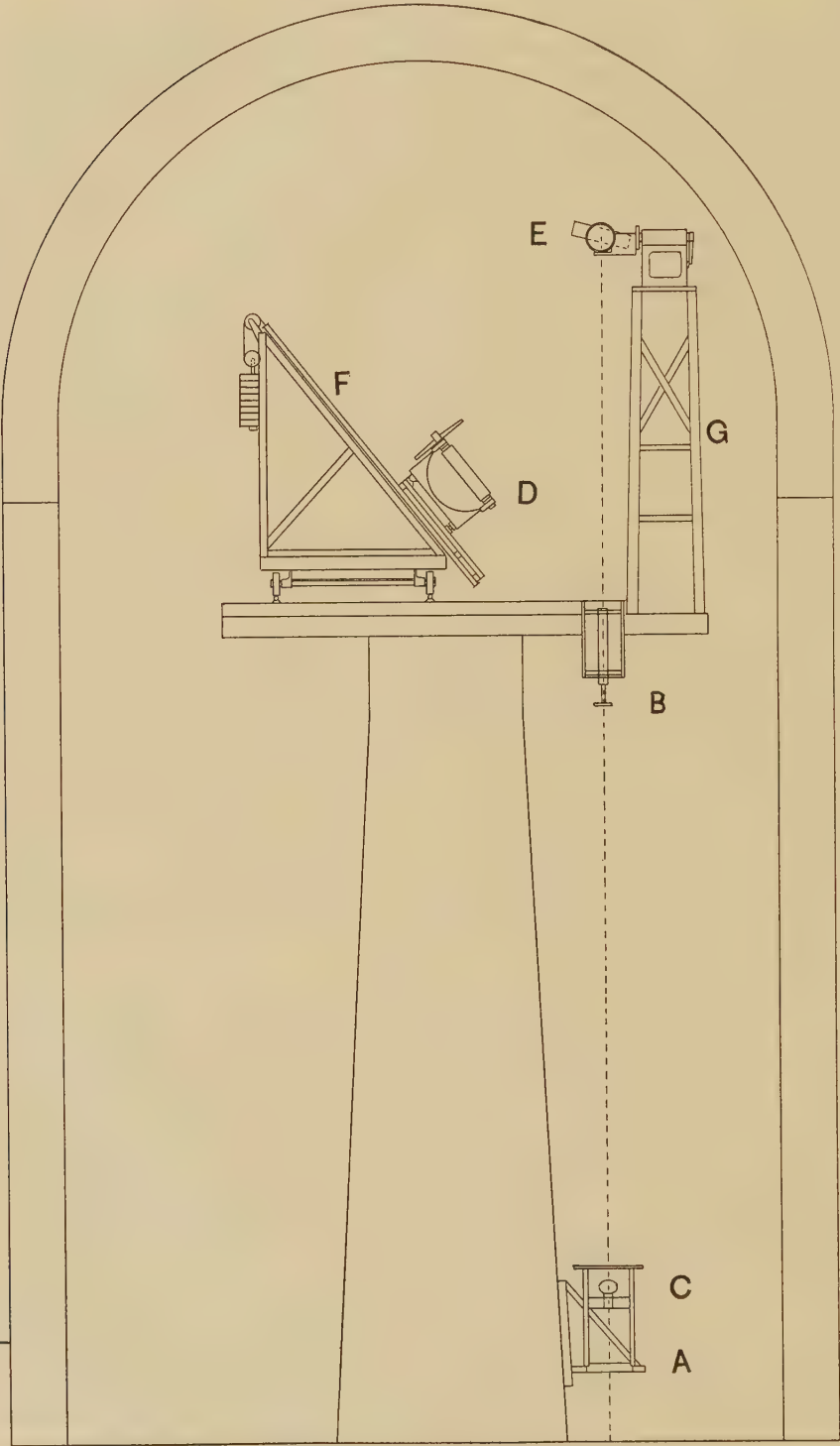


FIG. 1.—*Diagrammatic Sketch of Telescope in Tower.*
(Scale, 1 inch, approximately 5 feet.)

Fig. 2



Fig. 3



Photographs of a portion of upper platform (above) and of
main mirror and Coudé.

H. H. Plaskett, The Oxford Solar Telescope and Hartmann Tests of its Performance.

the Cassegrain towards the main mirror meets a small flat, C, hereafter referred to as the Coudé, of 14 cm. ($5\frac{1}{2}$ inches) clear aperture ; this turns the beam off to the west, and the resulting image is formed on the slit of a large spectroscope, the body of which lies in a long tunnel running west through the main building. The fused silica discs of the Cassegrain and Coudé mirrors are each 1.8 cm. (0.72 inch) thick. Vertically above the Cassegrain telescope lies the secondary flat, E ; this reflects the light received from the cœlost, D, down into the telescope. These two mirrors are fused silica discs of 40.6 cm. (16 inches) clear aperture and 7 cm. ($2\frac{3}{4}$ inches) thickness.

The cœlost and secondary mirror, as can be seen from figs. 1 and 2, are mounted on a steel platform approximately 4 m. (13 feet) square. This latter is carried on two I beams (6 by 3 inches) grouted into the concrete cap on top of the pier, and in addition is braced by six tubular ties from the body of the pier (not shown in fig. 1). Following Newall's design for the McClean Solar Telescope,* the cœlost is mounted on a carriage, F, on which it can move, parallel to the axis of the Earth, to allow for the varying declination of the Sun ; this carriage, as is customary, may be moved on rails east and west to avoid the shadow cast by the secondary. The cœlost proper, whose axis may be adjusted in altitude and azimuth, is driven through a worm and worm-wheel by a synchronous motor run off the mains ; a differential gear in the drive, controlled by push-button switches in the focal plane, allows of a guiding motion at slightly less than the diurnal rate. The secondary mirror is mounted altazimuthly on the top of a small tower, G, at the south of the steel platform ; a guiding motion in declination at the diurnal rate can be given to the image by rotating this mirror through a worm and worm-wheel by a D.C. motor, also controlled by push-button switches in the same keyboard as the cœlost controls. The tower, G, also carries a hoist, by means of which the 16-inch flats can be lowered to the floor below for silvering. A vertical steel ladder gives access to the platform ; the observer must ascend this in order to open the dome and make the preliminary setting of the cœlost and secondary. Apart from a possible subsequent motion of the cœlost, east or west, guiding and dome motion can all be effected from the bottom of the tower.

The main mirror, A, and the Coudé, C, are mounted in cells in a carriage, (fig. 3). This carriage has adjustments for levelling, rotation and centring, and is bolted to a braced bracket grouted into the bottom of the pier on the south side ; the Coudé, held in position by four thin webs, has an independent lateral adjustment. Both mirror cells may be tilted by squaring-on screws of the concentric type. The Cassegrain mirror, B, is mounted on a shaft, the bearings for which are held by webs in a carriage mounted on the steel platform ; this carriage has adjustments for levelling, rotation and centring, while an independent adjustment in the webs make it possible to centre the shaft carrying the mirror. The mirror cell may be tilted on its shaft by squaring-on screws. The shaft may be moved vertically up and down in its bearings by means of a screw, driven by a D.C. motor,

* *Annals of the Solar Physics Obs.*, 3, Part 1, 1915.

and controlled by push-button switches in the hand keyboard below. The resulting range of motion of the focal plane is 1.8 m. (6 feet), the change of focus occurring at the rate of 2.6 cm. per second.

Considerable and very necessary economies in construction have been effected by the use, wherever possible, of fabricated parts in place of castings. In the search for economy, however, no adjustments necessary for the efficient operation of the telescope have been omitted, while the detail of the design and the finish of the mechanical parts are in accordance with the high standards of Grubb, Parsons.

Section 2.—Preliminaries to and Reduction of the First Hartmann Test

Desirability of a Hartmann Test.—It has not hitherto been customary to test the whole optical train of a solar telescope as a unit. This omission has arisen partly from the circumstance that such a test is not carried out as easily, either in the optical shop or in the observatory, as the corresponding test for a stellar telescope, but more particularly because it is generally assumed that a system composed of elements, each of which is satisfactory within some customary tolerance, will suffice to give optical definition superior to that afforded by solar "seeing." This view, however, neglects the fact that when the best seeing occurs, that is near the rising and setting of the Sun in the summer, the tower telescope is then used under the most exacting conditions of high inclinations of the cœlostæt and secondary flats; there is, accordingly, no assurance that the limiting definition then obtained is not imposed as much by the telescope as by the seeing conditions. This view also neglects the fact that in a long optical train the cumulative effect of tolerances, ordinarily adequate for the surfaces used alone, may produce a total aberration far from negligible. In this connection it must be remembered that the two large flats, at best not easy to figure, are ordinarily tested in the optical workshop in conjunction with an auxiliary sphere; * it is clearly desirable to have a completely independent test of their performance in conjunction with the system of which they are to form an integral part. In short, the grounds offered for not making a test appear to be inadequate, and, indeed, until it is precisely known from a test what the limitations of the telescope are, it would appear difficult either to use the instrument to full advantage or to be sure that the observed "poor solar seeing" is not in part introduced by the telescope itself.

Given that a test of the system as a whole is desirable, if not necessary, there can be little doubt that Hartmann's method affords the simplest and most accurate quantitative way of making that test. The method is, however, sometimes criticized by the working optician who, especially for tests at the centre of curvature of paraboloids, finds it neither as rapid nor as accurate as the Foucault knife-edge test. The first of these criticisms is just, so that the Hartmann test has properly little or no place during figuring operations; but what the test loses in speed it must gain in accuracy, because

* G. W. Ritchey, *The Modern Reflecting Telescope*, *Smithsonian Contributions to Knowledge*, 34, 25, 1904.

it is completely free from all subjective influence in measurement. Indeed it is difficult to see how the two methods can be even comparable in accuracy, because though both depend upon geometrical ray-tracing from the surface, in the Foucault test the measurement depends upon a difficult photometric match in an unsuitable field, while in the Hartmann test measurement depends upon a simple distance determination only. In this connection it is not without significance that the final test of the exceptionally good Forcalquier reflector was carried out at the centre of curvature by its maker using Hartmann's method.* Criticism is sometimes offered of both tests on quite other grounds, namely, that they measure the aberrations of the system in terms of geometrical rays rather than in terms of phase differences at the image. However, as L. C. Martin † has pointed out, all tests fall into one or other of these two categories, and results obtained from the one are, generally speaking, immediately transferable into the other. Thus the results of the Hartmann tests about to be described are all expressed in terms of path differences, and the so-called Hartmann criterion plays no part in assessing the performance of the system.

In the light of these considerations a Hartmann test of the telescope has from the first appeared desirable. Tests with an artificial star in the Grubb, Parsons testing tunnel and at the observatory having proved impossible in the one case and unsatisfactory in the other, plates of an actual star were first secured in 1936 November. Following a brief description of the methods of collimating the system and of finding the star, the balance of this section will be concerned with the results from this test.

Collimation.—A plumb bob hung from the centre of the Cassegrain mirror-shaft makes it possible to place the centres of it, the main mirror and the Coudé, on a common vertical axis. The plumb bob having been removed, a small viewing telescope is then set up in the focal plane of the solar telescope, and adjusted until the centre of the Coudé, marked by a pair of wires stretched across it, appears on the cross-wires of the viewing telescope. The cell of the Coudé is then adjusted by its squaring-on screws until the image of the Cassegrain mirror and its supporting webs, seen in the Coudé, are also centred on the cross-wires of the viewing telescope. Next, the Cassegrain mirror is squared on, so that the rim of the main mirror is seen centred in the Cassegrain. Finally, the main mirror is tilted by its squaring-on screws until the image of the Cassegrain webs, seen in this mirror after reflection by the Cassegrain, line up with the Cassegrain webs themselves, both of course as seen in the viewing telescope after reflection by the Coudé.

Finding the Star.—Since the telescope has no circles, and since there are four disposable motions of the cœlostæt and secondary mirrors, it is at first sight not a simple matter for an observer working alone to pick up a given star and to centre it so that the paraboloid is filled with light. The following procedure has, however, proved convenient and fairly rapid. First the cœlostæt is placed due south of the secondary, and adjusted in

* A. Danjon and A. Couder, *Lunettes et Télescopes*, p. 507, 1935.

† L. C. Martin, *Introduction to Applied Optics*, 2, 248, 1932.

hour angle and declination until an image of the desired star is seen by the observer, his head being in front of the secondary. Next, the secondary is adjusted until the image can be seen by the observer when his head is vertically over the telescope proper. At this stage it is desirable, though not necessary, while still retaining the star image so to move the mirrors that the star is seen centred in the cœlostæt and the cœlostæt centred in the secondary. The observer goes next to the bottom of the tower, where he sees the star image now reflected by the Coudé; by means of the guiding buttons he moves this image until the much brighter image formed by reflection from the main mirror and the Cassegrain is seen centred in the Coudé. Looking now through the viewing telescope, left in position after the collimation, the paraboloid is seen illuminated by the light of the star; the star image proper can of course be seen and centred on the cross-wires by removing the objective of the viewing telescope and using the eyepiece alone. If the paraboloid is not filled with light, instead of a circular extra focal patch with a central shadow due to Cassegrain and Coudé there will be seen an annular area with a shadow across one side. If this shadow is to the north or south this means that the cœlostæt must be moved in declination; fortunately this can be done without losing the star image. If the shadow is to the east or west, the secondary must be rotated about its north-south axis; this in turn requires a compensating motion of the cœlostæt in hour angle, and the process of bringing the star image down to the focal plane must be repeated. A few trials and about two hours' work serve as a rule to complete the process; on any subsequent night, if the mirrors are left in position, a motion of the cœlostæt in hour angle serves to pick up a fully centred image of the star.

Result of First Test.—The first plates for this test were secured on the nights of 1936 November 9 and 14. The Hartmann diaphragm, mounted on the main mirror, consisted of 38 mm. diameter holes, cut in stout drawing-paper and arranged in three zones of radii 95.2, 120.5 and 145.8 mm. at different azimuths; the absence of central zones was of course due to the shadow of the Cassegrain. Hartmann* recommends a minimum diameter for the holes of $1/400$ th of the focal length, but the present diaphragm with holes $1/600$ th has been found to give well-defined patches free from diffraction effects. The photographic plate was held in position normal to the optical axis on a suitable wooden stand mounted on a pier in the focal plane; the separation of the two positions, inside and outside the focus, was 1090 mm. Various fast plates were used, and depending upon their speed and the transparency of the sky exposures ranged between 20 and 60 seconds; in spite of the difference between solar and sidereal rates and the absence of guiding, a number of well-defined plates were obtained, of which the four best were selected for measurement.

Each plate was measured in two positions, direct and reversed, on the Hilger Measuring Micrometer,† and the mean of the measured separation of the extra focal patches for each zone and azimuth converted into focal

* J. Hartmann, *Zs. f. Inst.*, **24**, 5, 1904.

† Loaned by the Government Grant Committee of the Royal Society.

settings.* These settings are given in Table I, the first and second columns containing the radius of the zone and the azimuth, the latter measured from the north through the east, while the next four main columns contain the focal settings for the four plates. These focal settings are expressed as residuals from the mean setting, weighted according to the radius—that is the area—of the zone. Focal settings for each plate are given in two separate columns, the first containing the setting for each azimuth in each zone, and the second the mean for the zone; since these means are formed from azimuths 90° apart, they are independent of astigmatism,* and show the effects of zonal aberration alone.

TABLE I
Values of $(f-f_0)$, 1936 November 9-14

y	θ	Plate No. 4		No. 5		No. 10		No. 11	
		mm.	mm.	mm.	mm.	mm.	mm.	mm.	mm.
95.2	0	-14.3	-9.0	-15.9	-11.3	-9.9	-6.0	-12.9	-10.0
	90	-3.7		-6.7		-2.0		-7.1	
120.5	45	+5.3	+9.3	+6.1	+9.6	+1.3	+7.6	+2.8	+8.5
	135	+13.4		+13.1		+13.8		+14.3	
145.8	22.5	-5.5	-1.9	-5.3	-0.5	-9.4	-2.4	-7.8	-0.5
	67.5	-4.5		-2.2		-5.7		-3.0	
	112.5	+4.2		+6.5		+6.8		+9.4	
	157.5	-1.6		-0.9		-1.5		-0.6	

Considering the zonal aberration first, Danjon and Couder† have proposed two criteria which must be simultaneously satisfied if a telescope is to be regarded as “satisfactory.” First, the transverse aberration for each zone must be comparable with the radius of the theoretical diffraction disc; and secondly, the path difference of the light coming from the different zones must not exceed the Rayleigh limit of $\lambda/4$. The values of the transverse aberration, T , at the chosen focal plane, f_0 , are given in Table II,

TABLE II
Values of T/ρ

y	$f-f_0$	T	$ T/\rho $
mm.	mm.	mm.	
95.2	-9.1	-0.0388	0.81
120.5	+8.8	+0.0475	.99
145.8	-1.3	-0.0085	.18

* J. Hartmann, *Zs. f. Inst.*, **24**, 5, 1904.

† A. Danjon and A. Couder, *Lunettes et Télescopes*, p. 522, 1935.

where the first column gives the radius of the zone, the second $(f-f_0)$ for the zone, being the mean of the results in the second columns of the four main columns of Table I, and the third column gives $T=y(f-f_0)/f_0$, where $f_0=2.231 \times 10^4$ mm. The radius, ρ , of the first dark ring of the Airy diffraction disc is given by $1.22\lambda f_0/d$, where d is the diameter of the mirror or objective; choosing $\lambda=0.56 \mu$, we find for the present telescope of aperture ratio $f/70$, $\rho=0.048$ mm. In the final column of Table II are given the values of T/ρ , and it will be seen that they amply satisfy Danjon and Couder's first criterion.

Coming to the second criterion we note, in accordance with elementary theory,* that the increase of path between the actual wave and an ideal spherical wave centred on the chosen focal point, per unit increase of height on the mirror, is approximately given by

$$\frac{dp}{dy} = \frac{T}{f_0}.$$

Integrating from the innermost zone of height y_1 , we find the path difference to any other zone of height y given by

$$p = \frac{1}{f_0} \int_{y_1}^y T dy. \quad (1)$$

Plotting T as a function of y from Table II and integrating graphically it is found that at $y=105, 115, 125, 135$ and 145.8 mm. respectively, the path differences $p = -0.008, +0.001, +0.021, +0.036$ and $+0.040 \mu$. Choosing $\lambda=0.56 \mu$, it follows that the maximum path difference is $\lambda/14$. If we displace f_0 back by 3.3 mm. so as to make the path difference between the innermost and outermost zones zero, we obtain an even more satisfactory image with the maximum path difference $\lambda/40$. In short, so far as zonal aberration is concerned, both criteria are amply satisfied and the telescope must be regarded as good.

When we come to consider the astigmatism the situation is not so satisfactory. In Table III are given for each zone at the observed azimuths the mean astigmatic difference of focus. Taking the innermost zone, $y=95.2$ mm., as an example, we find from Table I that the mean of $(f-f_0)$

TABLE III
Astigmatic Differences of Focus

y	$\theta=0^\circ$	$=22^\circ.5$	$=45^\circ$	$=67^\circ.5$	$=90^\circ$	$=112^\circ.5$	$=135^\circ$	$=157^\circ.5$
mm.	mm.	mm.	mm.	mm.	mm.	mm.	mm.	mm.
95.2	-4.1				+4.2			
120.5			-4.9				+4.8	
145.8		-5.7		-2.5		+8.0		+0.1

* L. C. Martin, *Introduction to Applied Optics*, I, 117, 1930.

at $\theta = 0^\circ$ is -13.2 mm., and at $\theta = 90^\circ$ is -4.9 ; expressing these as residuals from the mean focal setting for the zone, f_y , where $(f_y - f_0) = -9.1$, we obtain $(f - f_y)$ at $0^\circ = -4.1$ and at $90^\circ = 4.2$. These entries form the first line of Table III; the second and third lines, referring to the median and

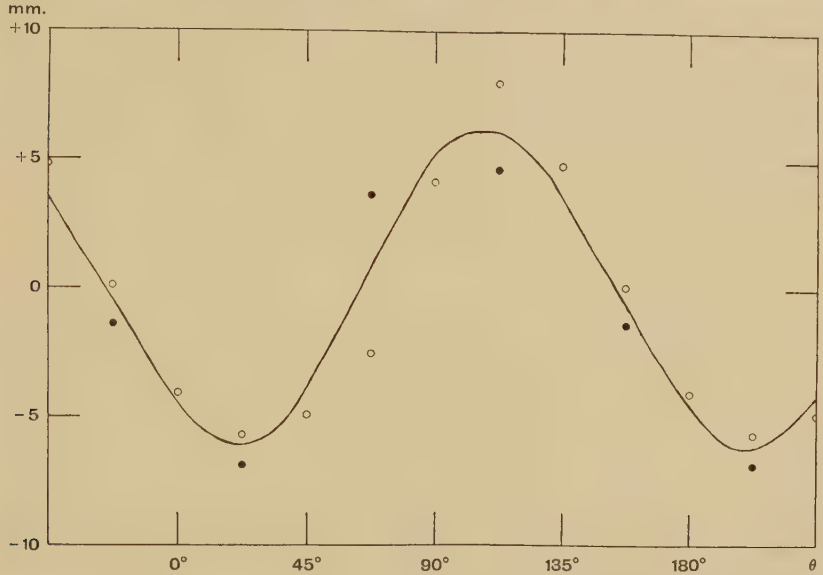


FIG. 4.—Values of $(f - f_y)_\theta$ as a function of azimuth (θ) for the first test. Open circles—Table III (1936 November 9, 14); filled circles—Table IV (1937 May 31, June 2).

outermost zones, are similarly obtained from Table I. Table III therefore contains the differences of focus due to astigmatism alone, the zonal aberration having been removed. These results are shown graphically in fig. 4, where the abscissa is the azimuth and the ordinate is $(f - f_y)$; note that $(f - f_y)_{\theta + 180^\circ} = (f - f_y)_\theta$. It is clear from the figure that astigmatism is sensibly present. As an additional check, however, some further Hartmann plates of Arcturus, with a new diaphragm having holes on the one zone $y = 121$ mm., were obtained in 1937 June. The three best of these were measured, and the values of $(f - f_y)_\theta$ for them are given in Table IV, while

TABLE IV
Values of $(f - f_y)_\theta$, 1937 May 31, June 2

θ	Plate 1	3	8	Mean
°	mm.	mm.	mm.	mm.
22.5	-7.8	-5.5	-7.3	-6.9
67.5	+5.0	+5.5	+0.5	+3.7
112.5	+6.2	+2.5	+5.3	+4.7
157.5	-3.5	-2.5	+1.6	-1.4

the means given in the last column are plotted in fig. 4. It will be seen that these new plates agree with the ones of Altair within the accidental error; it will also be noted, incidentally, that in view of the very small aperture ratio of the telescope the accidental error is not very large.

In order to find the path differences corresponding to the astigmatism shown in fig. 4, let r_1 and r_2 be the radii of curvature of the two principal sections of the wave surface, and let h_1 and h_2 be the corresponding perpendicular distances from a chord of length $2y$ to the wave surface. Then in the limit $r_1 = y^2/2h_1$ and $r_2 = y^2/2h_2$. But the path difference at zone y for the paraxial waves we are considering will be

$$p = h_2 - h_1 = \frac{y^2}{2} \left(\frac{1}{r_2} - \frac{1}{r_1} \right) = \frac{y^2}{2} \cdot \frac{a}{f^2}, \quad (2)$$

where $a = r_1 - r_2$ is the astigmatic difference of focus and $f = 22.31$ m. is the geometric mean of r_1 and r_2 . From fig. 4 $a = 13$ mm., and consequently from eqn. (2) we find the path differences at the three zones to be 0.21λ , 0.34λ and 0.50λ , where $\lambda = 0.56 \mu$. For the outermost zone the path difference has exceeded Rayleigh's limit, but if this were the maximum effect it would probably be regarded as a tolerable aberration for a solar telescope. Assuming, however, as is not improbable, that this astigmatism arises from one of the flats having a radius of curvature r and being used at angle of incidence i , then Rayleigh has shown that the astigmatic difference of focus is given by *

$$\frac{1}{f_1} - \frac{1}{f_2} = \frac{2}{r} \left(\frac{1}{\cos i} - \cos i \right), \quad (3)$$

where f_1 and f_2 are the foci in the plane of incidence and normal to it. The corresponding path difference will be

$$p = \frac{y^2}{r} \left(\frac{1}{\cos i} - \cos i \right). \quad (4)$$

If i for the present tests, which give $p = 0.5 \lambda$, is 25° (roughly the mean for the three flats), then at $i = 60^\circ$ (maximum for cœlost at rising summer Sun) the path difference will be 3.8λ . This would be a serious amount of astigmatism and would render the telescope of little value in the early morning hours when solar "seeing" is generally at its best.

As a result of these tests the three flats—the cœlost, the secondary and the Coudé—were taken back by Grubb, Parsons for re-examination at the end of 1937 June. They found the two large mirrors up to their standard, and doubted whether they could improve them by further figuring. The small Coudé, however, originally of the same quality, was now found to be distorted with a narrow cylindrical valley running across an otherwise slightly convex surface. On re-examination the Thermal Syndicate Limited found the disc—one of the earliest they had made by a process almost immediately superseded—to be defective, and they replaced it with a new one.

* Lord Rayleigh, *Collected Papers*, I, 437, 1880.

This was figured by the late J. A. Armstrong of Grubb, Parsons, and on subsequent test in the Hilger Interferometer was found to be flat to $\lambda/10$.

Section 3.—The Final Tests and the Probable Origin of the Remaining Astigmatism

Result of the Second Test.—The new Coudé having been put in place, the large mirrors resilvered and the telescope collimated, two sets of Hartmann plates were made with the single-zone diaphragm. The first set, somewhat under-exposed, was made of Altair on 1937 October 18 and 19; the three best plates were measured, and the values for these with their mean are given in Table V. The means are plotted in fig. 5 (p. 232)

TABLE V

Values of $(f-f_y)_\theta$, Test II (Altair), 1937 October 18, 19

θ	Plate 8	15	16	Mean
°	mm.	mm.	mm.	mm.
22.5	+ 9.4	+ 7.7	+ 5.8	+ 7.6
67.5	+ 7.4	+ 3.0	+ 2.2	+ 4.2
112.5	- 5.4	- 2.6	+ 0.2	- 2.6
157.5	- 11.4	- 8.1	- 8.2	- 9.2

with open circles, and combining with them the results from the second set of plates, to be described in a moment, it will be seen that the astigmatism is still present with a now equal to 18 mm. and with the azimuth of minimum focus shifted from $\theta = 22^\circ.5$, as in fig. 4, to $\theta = 135^\circ$. It is somewhat disappointing that the replacement of the defective Coudé by the new one has increased the astigmatism for the median zone from 0.34λ to 0.46λ ; since the new flat from its test in the interferometer must introduce negligible aberration, the interpretation of this result is, however, clear. The astigmatism observed in the first test must have been the sum of that introduced by the defective Coudé, plus the residual astigmatism in the telescope now observed in this second test. Since the sum is less than the residual astigmatism alone, it seems likely that the orientation of the defective Coudé must have been such that its principal section of maximum curvature coincided with the principal section of minimum curvature of the residual astigmatism. If this were the case, the astigmatism introduced by the defective Coudé alone must have corresponded to a path difference of 0.8λ ($0.8 - 0.46 = 0.34$), and with other orientations of this mirror astigmatism up to 1.26λ might have been obtained. The new Coudé eliminates this possibility.

In order to find the source of this residual astigmatism three possibilities remain. It may be due to a cylindrical deformation of the main mirror

(paraboloid), or of the Cassegrain, or, finally, to some slight curvature in one or both of the two large flats. In order to examine the first possibility the main mirror was rotated through 90° , the telescope recollimated and

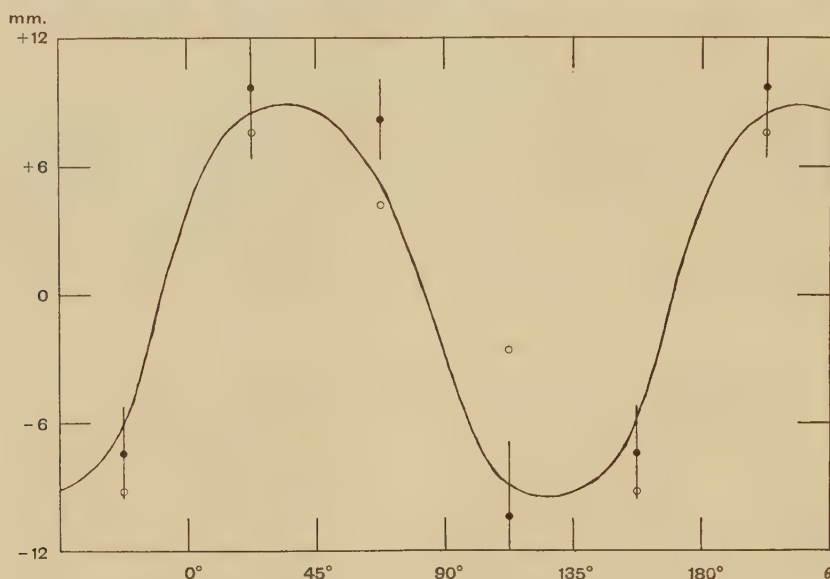


FIG. 5.—Values of $(f - f_y)_\theta$ as a function of azimuth (θ) for the second test. Open circles—Table V (1937 October 18, 19); filled circles with vertical line—Table VI (1937 December 30). Length of vertical line is standard error of a single plate.

a second set of Hartmann plates made with the same diaphragm on 1937 December 5 and 30, using the star Procyon. The resulting values of $(f - f_y)$ are given in Table VI; for convenience of later discussion the second

TABLE VI

Values of $(f - f_y)_\theta$, Test II (Procyon), 1937

θ	Dec. 5 $i_1 = 25^\circ.2$ $\theta_1 = 78^\circ.7$ E.	Dec. 30 $i_1 = 12^\circ.0$ $\theta_1 = 63^\circ.6$ W.
$^\circ$	mm.	mm.
22.5	+ 11.4	+ 9.7 $\pm$ 3.3
67.5	+ 6.3	+ 8.2 $\pm$ 1.9
112.5	- 9.6	- 10.4 $\pm$ 3.5
157.5	- 8.1	- 7.4 $\pm$ 2.2

column gives the values for the single good plate of December 5, while the third column gives the mean, with the standard deviation for a single plate, of the six good plates of December 30. The column headings contain i_1 , the angle of inclination for the cœlostát, and θ_1 , the azimuth

of the plane of incidence for the cœlost, measured from its meridian ; these are obtained from the expressions

$$\cos i_1 = \cos \delta \cos H/2 ; \quad \sin \theta_1 = \operatorname{cosec} i \sin H/2, \quad (5)$$

valid when the cœlost directs its reflected beam due south. The values given at the head of the third column are of course means, the dispersion in the individual values having been $3^\circ 18'$ for i_1 and $8^\circ 42'$ for θ_1 . It is readily seen that the corresponding quantities for the secondary mirror are given by $i_2 = (\phi + \delta)/2$, $\theta_2 = 0$, and are therefore constant for Table VI. The means from the third column of the table are plotted as filled circles in fig. 5 ; since within the accidental error they agree with the values from Altair (open circles), and in particular do not show the minimum focus displaced 90° clockwise to $\theta = 225^\circ$, we conclude that the residual astigmatism does not originate in the paraboloid.

In order to examine the second possibility, deformation of the $f/70$ Cassegrain mirror, advantage was taken of the fact that, in consequence of changes in the large spectrocope during the course of construction, it would be necessary in future to use the solar telescope with an aperture ratio of $f/62$. A new Cassegrain mirror of this aperture ratio, to be made of fused silica, was accordingly ordered from Grubb, Parsons ; it was figured by J. A. Armstrong using Hindle's method * and given a clear aperture of 14.6 cm. (5.75 in.). This mirror, mounted in a slightly remodelled Cassegrain cage, was installed in the telescope in 1938 July ; the equivalent focal length of the telescope was thus reduced from 22.31 m. to 19.80 m. (65 feet), and the scale correspondingly from 1" of arc = 0.108 mm. to 1" = 0.096 mm.

Result of the Third Test.—The final set of Hartmann plates was made of Altair on 1938 August 17 and 19. Apart from the presence of the new $f/62$ Cassegrain, the telescope was the same as when the Procyon plates of the second test were made ; in particular, the orientations of the two large flats in their cells were unchanged since resilvering had not been necessary. The plates were made with two diaphragms—one the single-zone diaphragm already used, and the other a new diaphragm with 38 mm.

TABLE VII
Values of $(f - f_y)_\theta$, Test III, 1938 August 17, 19

θ	Plate I $i_1 = 12^\circ.4$ $\theta_1 = 46^\circ.0$ E.	2 $i_1 = 9^\circ.5$ $\theta_1 = 24^\circ.1$ E.	17 $i_1 = 12^\circ.1$ $\theta_1 = 44^\circ.7$ W.	Means
0	mm.	mm.	mm.	mm.
22.5	+6.5	+6.1	+7.8	+6.8
67.5	+6.0	+4.6	+0.1	+3.2
112.5	-5.7	-4.4	-6.0	-5.4
157.5	-6.8	-6.3	-2.0	-5.0

* J. H. Hindle, *M.N.*, 91, 592, 1931.

holes arranged in various azimuths on three zones of radii, $y = 103.9, 122.8$ and 141.8 mm. The three best plates with the single-zone diaphragm were measured, and the corresponding values of $f - f_y$ are given in Table VII ; the results from the measured five best plates with the three-zone diaphragm are given in Table VIII, the two halves of the table referring to different orientations of the diaphragm on the main mirror.

TABLE VIII
Values of $f - f_0$, Test III, 1938 August 19

y	θ	Plate 10 $i_1 = 10^{\circ}.0$ $\theta_1 = 30^{\circ}.8$ E.	11 $i_1 = 9^{\circ}.3$ $\theta_1 = 21^{\circ}.2$ E.	Means	
		mm.	mm.	mm.	mm.
103.9	345	+2.8	+3.4	+3.1	+5.0
	75	+6.3	+7.4	+6.9	
122.8	15	+0.1	-0.1	± 0.0	-3.4
	105	-6.9	-6.9	-6.9	
141.8	45	+8.0	+6.3	+7.2	-0.7
	135	-8.9	-8.2	-8.5	

y	θ	Plate 13 $i_1 = 8^{\circ}.7$ $\theta_1 = 0^{\circ}.9$ W.	14 $i_1 = 9^{\circ}.0$ $\theta_1 = 16^{\circ}.2$ W.	16 $i_1 = 10^{\circ}.4$ $\theta_1 = 33^{\circ}.6$ W.	Means	
		mm.	mm.	mm.	mm.	mm.
103.9	300	-2.3	-0.8	-1.2	-1.5	+6.0
	30	+14.8	+14.8	+11.0	+13.6	
122.8	330	-8.4	-8.8	-7.4	-8.2	-2.1
	60	+3.6	+4.6	+3.5	+3.9	
141.8	0	+2.6	+0.2	-0.1	+0.9	-2.6
	90	-7.7	-6.8	-3.7	-6.1	

Considering the zonal aberration first, Table IX contains the mean values of $f - f_0$ from Table VIII, equal weight being given to the two orientations of the diaphragm. The remaining columns of the table contain the transverse aberration, T , and the ratio, T/ρ , computed as in the first test ; in the present case the focal length is only 19.80 m., and the radius of the Airy diffraction disc is 0.0424 mm. It will be seen that the ratios T/ρ amply satisfy Danjon and Couder's first criterion. Once again using eqn. (1) of the first test and plotting T as a function of y , a graphical integration yields the following path differences: $p = +.005, +.002, -.004, -.011$ and $-.016 \mu$ at $y = 111, 119, 127, 135$ and 141.8 mm. respectively. With $\lambda = 0.56 \mu$, the maximum path difference due to zonal aberration is therefore $\lambda/35$; if the focal plane is displaced some 1.3 mm. closer to the telescope,

TABLE IX
Values of T/ρ , Test III

y	$f-f_0$	T	$ T/\rho $
mm.	mm.	mm.	
103.9	+5.5	+0.0289	0.68
122.8	-2.8	-0.0172	.40
141.8	-1.6	-0.0117	.28

the even smaller maximum path difference of $\lambda/69$ is reached. As far as zonal aberration is concerned, the solar telescope with the $f/62$ Cassegrain satisfies Danjon and Couder's two criteria, and is perceptibly better than with the $f/70$ Cassegrain.*

Considering next the astigmatism, the values of $f-f_y$ from Tables VII and VIII are plotted in fig 6 (p. 236), the upper curve referring to plates 1, 2, 10 and 11 with a mean value of $\theta_1 = 30^\circ.6$ E., and the lower curve referring to plates 14, 16 and 17 with a mean θ_1 of $31^\circ.5$ W. The values of $f-f_y$ are of course obtained directly from Table VII, but from Table VIII they have to be derived in the same way as Table III was derived from Table I in the first test. It will be noticed that the two curves of fig. 6 are almost identical, placing the minimum focus at $\theta = 135^\circ$, as in the second test, and the astigmatic difference of focus, a , equal to 15 mm. Using eqn. (2) of the first test and recalling that $f = 19.80$ m., we find the path differences at $y = 103.9$, 121.8 and 141.8 mm. respectively to be 0.38, 0.52 and 0.68 λ , while the ratios of the radii of the discs of least confusion to the radius of the Airy disc are 0.96, 1.13 and 1.31.

Probable Origin of Residual Astigmatism.—Since the astigmatism is the same in direction and amount, within the accidental error, for the $f/62$ and the $f/70$ Cassegrains, it may be concluded that it does not originate in these mirrors. This leaves as the probable origin, since the main mirror has been excluded by the second test, some small curvature in one or both of the large flats. The nature of this curvature can be inferred, though with greater uncertainty, from the results of the second and third tests. Since the azimuth of minimum focus is 135° , that is 45° away from the plane of incidence of the secondary mirror, it follows from Rayleigh's investigation† that the astigmatism does not arise from a spherically concave or convex secondary.‡ Similarly, if it arose from a concave or convex cœlost,

* For what it is worth the Hartmann criterion for the telescope with the $f/62$ Cassegrain is 0.19, and with the $f/70$ Cassegrain is 0.26; Hartmann describes as "exceedingly good" anything with a criterion smaller than 0.5.

† Lord Rayleigh, *Collected Papers*, I, 437, 1884.

‡ This therefore excludes the possibility that the observed astigmatism arises from the flexure of the secondary under its own weight. That this is the case also follows from Couder's expression (*Bulletin Astronomique*, 7, 263, 1931) for the linear displacement at the centre of an edge-supported mirror, namely, $2.49 \times 10^{-9} R^4 / e^2$ cm.,

the azimuth of minimum focus would be expected to shift by some 60° as the azimuth, θ_1 , of the plane of incidence of this mirror changed from $30^\circ.6$ E. to $31^\circ.5$ W. Fig. 6, however, shows almost no change of the position of minimum focus, for these two azimuths of the plane of incidence of the cœlostât. If the astigmatism which has its origin in these two large

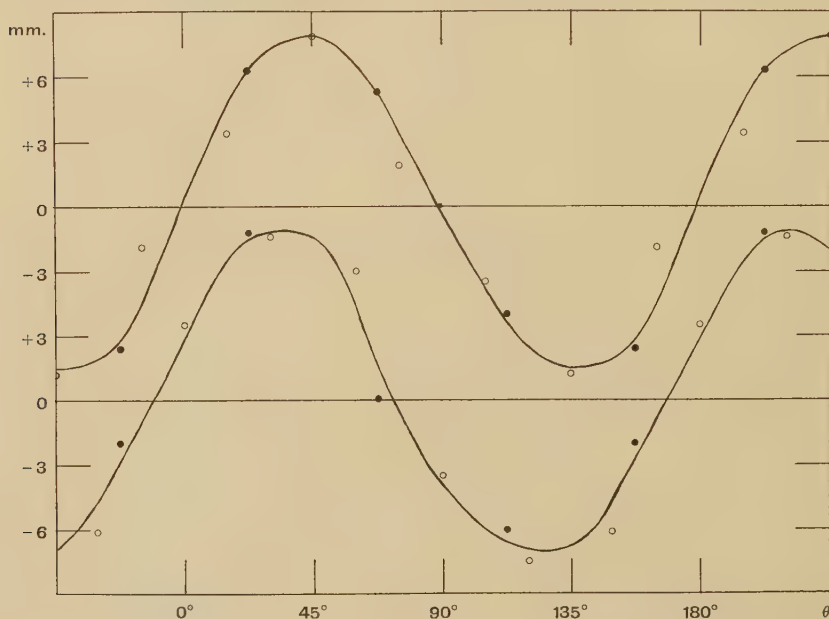


FIG. 6.—Values of $(f - f_0)_\theta$ as a function of azimuth (θ) for the third test. Open circles—Table VIII; filled circles—Table VII (1938 August 17, 19). The upper curve refers to plates taken when the plane of incidence of the cœlostât made a mean angle of $30^\circ.6$ E. with meridian, and the lower curve to plates taken when the corresponding angle was $31^\circ.5$ W.

flats does not arise from their sphericity, it must, surprising though it may appear, be due to one or both of the mirrors being cylindrical in shape.

The astigmatism of a cylindrical mirror is, however, insensitive to the angle of inclination at which it is used. A simple extension of Rayleigh's investigation* shows that for a spherical mirror the focus in an azimuth making an angle θ with the plane of incidence is given by

$$\frac{1}{f_\theta} = \frac{2}{r} \frac{\cos i}{\sin^2 \theta + \cos^2 \theta \cdot \cos^2 i}, \quad (6)$$

which, when $\theta = 0^\circ$ and 90° , leads directly to eqn. (3) of the first test. Now eqn. (6) also gives the power (reciprocal of the focal length) of a cylindrical

where R is the radius and e the thickness of the disc in cms.; for the present disc this displacement amounts to 0.086μ . The radius of curvature of the disc due to flexure is thus some 240 km., and at $i = 30^\circ$ (the angle of incidence at which it was used in making the tests) the resulting astigmatism for the solar telescope is from eqn. (4) only $\lambda/23$.

* Lord Rayleigh, *Collected Papers*, I, 437, 1884.



North-West Limb of Sun photographed at 9^h 16^m U T. on 1938 October 14.
Scale 1 mm. 3".5 of arc.

H. H. Plaskett, The Oxford Solar Telescope and Hartmann Tests of its Performance.

mirror in the azimuth perpendicular to the axis of the cylinder; in this case θ is the angle between this azimuth and the plane of incidence. In the azimuth at right angles to this—that is, in the axis of the cylinder—the power of the cylinder is zero. Consequently eqn. (6) gives the difference of power for a cylindrical mirror between the azimuths perpendicular and parallel to the cylindrical axis, where θ is still the angle between the azimuth perpendicular to the cylinder axis and the plane of incidence. This angle in the present case appears to lie between 30° and 60° , and it may then be shown from eqn. (6) that for a change of i from 0° to 60° the difference in power increases by 14 per cent. only for $\theta = 30^\circ$, decreases by 20 per cent. for $\theta = 45^\circ$ and by 38.5 per cent. for $\theta = 60^\circ$. Provided then we maintain the present orientation of the two large flats in their cells, and provided of course the astigmatism is really due to a cylindrical figure of one or both mirrors, it will be safe to use the cœlost at the maximum possible angles of inclination (for rising or setting Sun) without the astigmatic path difference exceeding 0.7λ or the circle of least confusion becoming sensibly greater than the Airy disc. In passing it may be noted that this result is confirmed by Table VI of the second test, where within the errors of observation the astigmatism for $i_1 = 25^\circ.2$ is the same as for $i_1 = 12^\circ$; if the astigmatism had arisen from a spherical cœlost the astigmatic difference of focus for 25° would have been four times greater than for $i_1 = 12^\circ$.

In brief, the Hartmann tests have led to the elimination of a defective mirror from the optical train. They have shown that the telescope in its final form has negligible zonal aberration, and astigmatism, which is tolerable for a solar telescope, certainly within the range of hour angles for which the tests have been performed, and possibly, if the astigmatism arises from slightly cylindrical flats, over the extreme range of hour angle. In the absence of Hartmann tests for other solar telescopes it is impossible to say how this instrument compares with them. In this connection visual estimates of solar definition, or photographs, permit almost no comparison, depending as they do upon the accidents of seeing at the time. For what it is worth, however, there is reproduced in Plate 6 a photograph of a portion of the solar disc, taken on Ilford Gas-Light Lantern plates, with the *full aperture* of the telescope and exposures of $1/150$ th of a second. Though the seeing conditions were not by any means the best possible at Oxford, the photograph appears to show nearly as much fine detail as those of Keenan with the 40-inch Yerkes refractor diaphragmed to 19 cm. aperture,* and of ten Bruggencate with the Potsdam solar telescope diaphragmed to 10 cm.† Irrespective of how this instrument compares with others, however, the Hartmann tests have served to show that it has ample reserves of optical definition for those exceptional moments of very good seeing upon which future investigations of the Sun must to a greater and greater extent depend.

* Philip C. Keenan, *Ap. J.*, **88**, 360, 1938.

† P. ten Bruggencate, *Zs. f. Ap.*, **16**, 374, 1938.

Summary

1. A new solar tower telescope is described, the image-forming part of which is a Cassegrain telescope of aperture ratio $f/62$ (in its final form) and of equivalent focal length of 19.8 m. (65 feet). It is built into one of the existing towers of the University Observatory.

2. All five mirrors of the telescope are discs of fused silica with a coefficient of expansion $1/20$ th that of crown and $1/7$ th that of pyrex glass. Mirror distortion due to the heat of the Sun should therefore be eliminated.

3. Evidence is advanced which suggests the desirability of making a Hartmann test of the whole optical train of a solar telescope as a unit. The method of collimating the telescope and of finding the star, necessary preliminaries to a Hartmann test, are described.

4. In all, three Hartmann tests, each with a slightly different optical train, were made. The first showed negligible zonal aberration and some astigmatism, the latter being traced to a small flat which had warped since the original figuring. For the second test this flat was replaced by one correct to $\lambda/10$; residual astigmatism was still present, however, and the test served to show that this must arise either from the Cassegrain mirror or the two large flats (the cœlostæt and secondary mirrors). The third test was made with a new Cassegrain, and showed that the residual astigmatism originated from a probably cylindrical shape of the two large flats. The net result of this final test was to show negligible zonal aberration and astigmatism with a path difference not in excess of 0.7λ , the latter probably almost independent of hour angle.

In conclusion I should like to express my gratitude to the University of Oxford for the grant which made the purchase of this instrument possible, and my very sincere thanks to Mr. Cyril Young of Grubb, Parsons for the detailed design of the instrument, and for his co-operation in its perfection—co-operation carried long past the stage where it was commercially profitable. Finally, I should like to acknowledge the interest and enthusiasm of Mr. George Neale of the observatory staff, and the help he has given at all stages in the design and erection of the telescope.

University Observatory, Oxford :
1938 December.

THE THEORY OF PRESSURE IONIZATION AND THE CONSTITUTION OF THE WHITE DWARF STARS.

F. C. Auluck.

(Communicated by Professor E. A. Milne)

1. The present paper is concerned with the following problem: What is the radius of a spherical aggregate of *cold** or degenerate matter (composed of atoms of one element only) of mass M in equilibrium under its own mutual gravitational forces, the degree of ionization in the material being treated as a function of the density in accordance with the theory of ionization in cold matter (pressure ionization)? The solution of the above problem when the degree of ionization is assumed to be independent of the density (*i.e.* the theory of pressure ionization is ignored) has been discussed in considerable detail by Milne † and also others ‡ and is now well known. The theory gives for the radius R and the mass M the expression

$$R = \frac{l}{\mu^{5/3}} \left(\frac{\odot}{M} \right)^{1/3}, \quad (1)$$

$$l = \frac{5(\omega_{3/2}^0)^{1/3} K}{2^{7/3} \pi^{2/3} G \odot^{1/3}} = 2.79 \times 10^9 \text{ cm.}, \quad (2)$$

and, further, the central density ρ_c and the mean density ρ_m are connected by the relation

$$\frac{\rho_c}{\rho_m} = \frac{(\sigma_{3/2})^{3/2}}{3(\omega_{3/2}^0)}, \quad (3)$$

where $\omega_{3/2}^0$ and $\sigma_{3/2}$ are constants characteristic of Emden's solution of Emden's equation of index $3/2$, G is the gravitational constant, K is what is known as the degenerate gas constant, $\odot$ is the mass of the Sun and μ is called the mean molecular weight per free electron, and its value is a measure of the degree of ionization of the material.

Before the theory of pressure ionization it has been usual to take for μ the value corresponding to complete ionization of the material. In case when the material is composed of atoms of one kind only, of say atomic weight A and atomic number Z , this will be equal to A/Z . In the sequel this value of μ equal to A/Z will be denoted by μ_0 . However, as the theory

* Matter will be referred to as *cold* or degenerate when any free electrons present constitute a degenerate gas in the sense of Fermi-Dirac statistics.

† Milne, *M.N.*, **92**, 610, 1932. The results quoted here are from Milne's paper with minor changes in notation explained in the text.

‡ Chandrasekhar, *M.N.*, **95**, 207, 1935.

of pressure ionization shows, the value of μ depends on the density of the material and therefore it is not justifiable

(1) to treat it as constant throughout the configuration, and

(2) to take it equal to the value corresponding to complete ionization.

Recently Kothari* has incorporated the theory of pressure ionization in relation (1) by taking for μ not the value for complete ionization, but the value as given by the theory of pressure ionization corresponding to a density equal to the mean density of the configuration. For M not much less than the mass of the Sun, Kothari finds that the (average) value of μ does not differ appreciably from μ_0 as used to be *assumed* before without any *a priori* justification, but for smaller masses the ionization is much less and the value of μ correspondingly larger than μ_0 . In relation (1), when μ is *assumed* independent of M , R increases continuously as M decreases. But, when combined with the theory of pressure ionization, (1) leads to the existence of a maximum radius for a cold body. The value of the maximum radius ($R_{\max}$) and the corresponding mass (M_0) are found to be of the same order of magnitude as the radius and mass of the planet Jupiter, which is the largest (in size) cold body known, and an agreement better than the order of magnitude should not be expected as the theory ignores all geophysical and geochemical complications and makes many other simplifying assumptions.†

As remarked above, Kothari has not taken account of the variation of μ inside the configuration, and it seems worth while to solve exactly the formal problem of the equilibrium configuration of a cold body *taking into account this variation of μ inside the configuration*. Our results, as would be expected, do not in general differ appreciably from those of Kothari. However, as the variation of μ inside the configuration is taken account of, the theory is carried a stage further. In section (2) the equations of the problem are derived, and section (3) deals with the discussion of results.

2. *Equations of the Problem and its Solution.*—If p is the pressure, ρ the density at a distance r from the centre, and $M(r)$ the mass enclosed within radius r , then, since for a cold body the radiation pressure is negligible compared to the gas pressure, we have for the equation of mechanical equilibrium

$$\frac{dp}{dr} = -\rho \frac{M(r)}{r^2} G, \quad (4)$$

and further ‡

$$\frac{d}{dr} M(r) = 4\pi r^2 \rho. \quad (5)$$

The pressure p in cold or degenerate matter is almost entirely due to the free electrons (the heavy particles making negligible contributions)

* Kothari, *M.N.*, 96, 833, 1936.

† A recent paper by Wildt (*Astrophysical Journal*, 87, 508, 1938) emphasizes the great importance of geophysical and geochemical considerations in a discussion of planetary constitution.

‡ This may be called the equation of the "conservation of mass."

and is given by the usual expression

$$p = \frac{K\rho^{5/3}}{\mu^{5/3}}, \quad (6)$$

where

$$K = \frac{8\pi}{15} \frac{h^2}{m} \left(\frac{3}{8\pi m_H} \right)^{5/3} \quad (7)$$

and μ is the mean molecular weight per free electron.

The theory of ionization in degenerate matter composed of atoms of one kind only has been worked out by Majumdar and Kothari, and Kothari.* Kothari has obtained on certain simplifying assumptions a relation between the degree of ionization and the density of the material. He finds that

$$\mu = \frac{\mu_0}{\left(1 - \frac{K'}{\rho^{1/3}}\right)^{3/5}}, \quad (8)$$

$$K' = (\Delta ZA)^{1/3} = \rho^{*1/3} \left(1 - \frac{1}{Z^{5/3}}\right), \quad (9)$$

$$\Delta = \left\{ 2 \cdot 3^{1/3} \cdot \pi^{2/3} \frac{me^2 m_H^{1/3}}{h^2} \right\}^3, \quad (10)$$

where m is the mass of the electron, e its charge, m_H the mass of the hydrogen atom and h Planck's constant. ρ^* denotes the density corresponding to single ionization (one free electron per ion) of the material (*i.e.* for $\rho = \rho^*$, $\mu = A$).

Introducing new variables z and x defined by

$$\rho = \left(\frac{4K'}{5} z \right)^3, \quad (11)$$

$$r = \frac{5}{4} \left\{ \frac{K}{\pi G K' \mu_0^{5/3}} \right\}^{1/2} x, \quad (12)$$

equations (4), (5) and (6) reduce to

$$\frac{1}{x^2} \frac{d}{dx} \left\{ x^2 (z-1) \frac{dz}{dx} \right\} + z^3 = 0, \quad (13)$$

$$M(r) = -4\pi \left\{ \frac{KK'}{\pi G \mu_0^{5/3}} \right\}^{3/2} x^2 (z-1) \frac{dz}{dx}. \quad (14)$$

Equation (13) is our fundamental equation.

Let ρ_c denote the central density, then, if z_0 is the value of z at the centre,

$$z_0 = \frac{5}{4K'} \rho_c^{1/3}. \quad (15)$$

* Kothari, *Proc. Roy. Soc., A*, **165**, 486, 1938. Other references are given there.

At the surface $p \rightarrow 0$, and therefore from equations (6) and (8) $\rho \rightarrow K'^3$ and hence $z \rightarrow \frac{5}{4}$. As the density increases monotonically from the surface inwards, this represents the minimum value of z .

We solve equation (13) with the initial conditions $x=0$, $z=z_0$ and $\frac{dz}{dx}=0$. The last condition follows from the fact that at the centre, the gravitational attraction being zero, $\frac{M(r)}{r^2} \rightarrow 0$ and hence $\frac{dz}{dx} \rightarrow 0$.

The radius R and mass M of the configuration will be given by

$$R = \frac{5}{4} \left\{ \frac{K}{\pi G K' \mu_0^{5/3}} \right\}^{1/2} x_1 \quad (16)$$

$$M = 4\pi \left\{ \frac{K K'}{\pi G \mu_0^{5/3}} \right\}^{3/2} v_1, \quad (17)$$

where x_1 is the value of x corresponding to $z = \frac{5}{4}$ (the boundary value) and v_1 is the value of $\left\{ -x^2(z-1) \frac{dz}{dx} \right\}$ for $x = x_1$.

It may be noted that in the special case when $\frac{z_0 - z}{z}$ is small, a power-series expansion can be obtained. Substituting $u = x^2$, equation (13) can be transformed to

$$\frac{d}{dz} \left\{ \frac{u(z-1)}{\frac{du}{dz}} \right\} + \frac{z-1}{2} + \frac{z^3}{4} \frac{du}{dz} = 0. \quad (18)$$

If $u = a_1(z_0 - z) + a_2(z_0 - z)^2 + a_3(z_0 - z)^3 + \dots$ we get

$$\begin{aligned} a_1 &= \frac{6(z_0 - 1)}{z_0^3}, & a_2 &= \frac{3(4z_0 - 9)}{5z_0^4}, \\ a_3 &= \frac{18(7z_0 - 47)}{175z_0^5}, & a_4 &= -\frac{56z_0^2 + 1391z_0 - 1507}{350(z_0 - 1)z_0^6}, \\ a_5 &= -\frac{198254z_0^3 + 705803z_0^2 - 2215768z_0 + 1285461}{336875z_0^7(z_0 - 1)^2}. \end{aligned}$$

This series is useful for calculating x_1 and v_1 for some * values of z_0 . Tables for $z_0 = 1000, 500, 100, 10, 4, 3, 2, 1.5$ are given in the appendix. In the calculations four-figure tables only were used.

3. We shall now, with the help of equations (16) and (17), express the results of the preceding sections in terms of M and R , the mass and the radius of the configuration. To do this we have to make some assumption about the chemical constitution of the material and the values of ρ^* —the density corresponding to single ionization. Following Kothari, two

* For $z_0 \leq 1.35$ the results are accurate up to four places of decimals.

assumptions have been made regarding the chemical composition of the material.*

(1) Assumption *F*, the matter is assumed to be iron ($A = 56$, $Z = 26$).

(2) Assumption *H*, the matter is assumed to be hydrogen ($A = 1$, $Z = 1$).

The value of K' is given by (9). However, as Kothari has remarked, in astrophysical applications it is better to identify ρ^* in (9) with the ordinary density of the normal metal rather than take the value as given by the theory (eqn. 9), which is based on a number of simplifying assumptions.†

In the case of iron, therefore, we shall take ρ^* to be 7.86 (the density of the normal metal). However, in the case of hydrogen we have to fall back on equation (9) ‡ and take for K' the value given by it.

The results of our calculations are shown in fig. 1, where $\log \frac{M}{\odot}$ is plotted against $\log R$.

The results calculated by us can be approximately fitted into the curve

$$ae^{-cy} = \cosh \left(b + \frac{cx}{3} \right), \quad (19)$$

where

$$y = \log_{10} R, \quad x = \log_{10} \frac{M}{\odot}, \quad c = \log_e 10.$$

On the assumption *F*,

$$a = 2.765 \times 10^9, \quad b = 1.9629,$$

and on the assumption *H*,

$$a = 1.245 \times 10^{10}, \quad b = 2.1889.$$

For a given value of M , the difference between the value of y given by (19) and that given by our calculations is less than .001, and increases from zero to this value as z_0 increases from 5/4 to about 3, and then again decreases to zero as z_0 increases. The difference between Kothari's results (which ignores the variation of μ inside the configuration) and our results cannot be depicted in fig. 1, being small.

* At the present stage of the theory the relation between μ and ρ for a mixture of elements is not known.

† For example, in the case of iron, equation (9) gives for ρ^* the value 69.9 gm./cm.⁻³, whereas for single ionization the density should not, obviously, be much different from the density of the normal metal (7.86 gm./cm.⁻³). The difference between the theoretical value and the density of the normal metal is rather large, but it may be added that this comparison with the density of the normal metal and ρ^* is rather unfair to the theory of pressure ionization, as the assumptions made in the theory render it particularly inaccurate in the region of low ionization.

‡ It is of interest to note that recently Sommerfeld and Walker (*Annalen der Physik*, 32, 56, 1938) have calculated the energy of the ground state of the hydrogen atom surrounded by an infinitely high and steep potential wall. They find that with decreasing radius of the wall the energy rises from its normal negative value and becomes positive when the radius is $\geq 1.835a_H$, a_H being the radius of the first Bohr orbit. From this it follows that for density exceeding $\rho = 0.44$ gm./cm.⁻³ hydrogen should be fully ionized. This is in reasonable accord with results obtained by Kothari, equation (8) showing about 70 per cent. ionization for $\rho = 0.44$ gm./cm.⁻³.

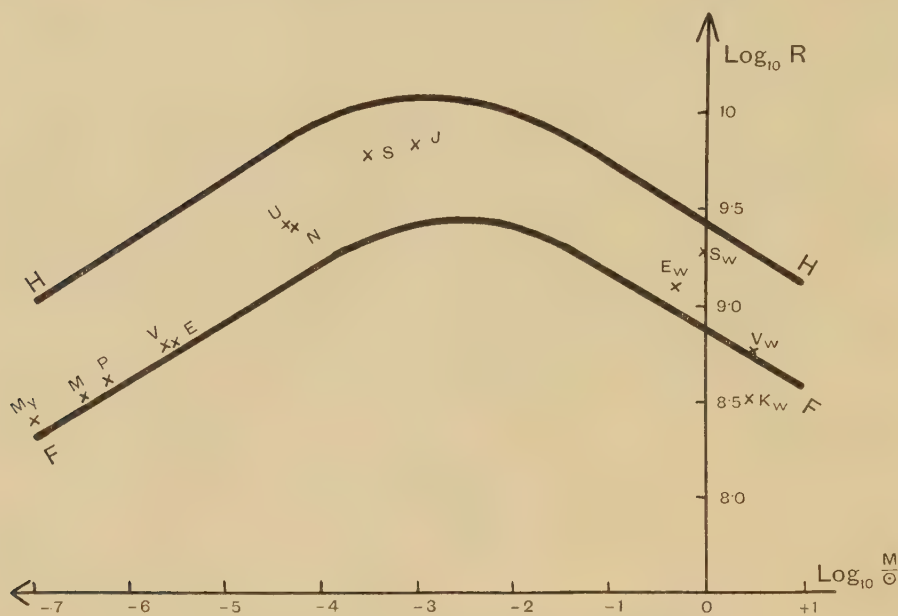


FIG. 1.—The curves show the theoretical relation between mass M and radius R . ($\odot$ = mass of the Sun.) The curve HH is for hydrogen and the curve FF is for iron.

M_Y = Mercury. M = Mars. S = Saturn. E_W = O_2 Eridani B.
 V = Venus. J = Jupiter. N = Neptune. S_W = Sirius B.
 E = Earth. U = Uranus. P = Pluto. K_W = A.C. 70° 8247.
 V_W = Van Maanen's star.

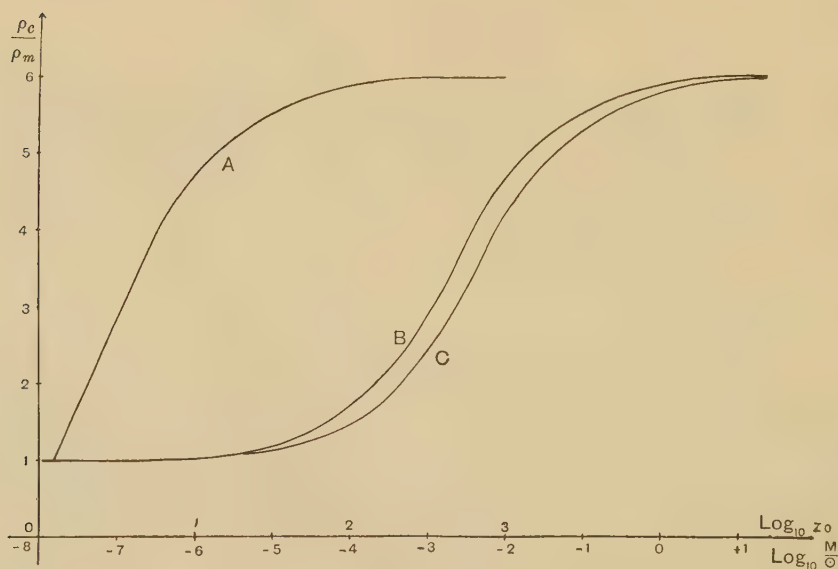


FIG. 2.

A —shows the relation between $\frac{\rho_c}{\rho_m}$ and $\log z_0$.
 B —shows the relation between $\frac{\rho_c}{\rho_m}$ and $\log \frac{M}{\odot}$ in the hydrogen case.
 C —shows the relation between $\frac{\rho_c}{\rho_m}$ and $\log \frac{M}{\odot}$ in the iron case.

Another point of interest may be noted :

$$\frac{\rho_c}{\rho_m} = \frac{(z_0 x_1)^3}{3 v_1} \tag{20}$$

It will be seen that $\frac{\rho_c}{\rho_m}$ is a function of z_0 , that is ρ_c , whereas when μ is treated constant throughout the configuration $\frac{\rho_c}{\rho_m}$ has a constant value 5.99 as given by the equation (3). When we treat μ as a function of ρ , $\frac{\rho_c}{\rho_m}$ increases from 1 to 5.99 as ρ_c increases from its minimum value K' to infinity. In fig. 2 the curve A shows the relation between $\frac{\rho_c}{\rho_m}$ and z_0 , and the curves marked B and C the relation between $\frac{\rho_c}{\rho_m}$ and M on the assumptions H and F respectively. It will be seen that for M_0 , that is the mass corresponding to the maximum radius, $\frac{\rho_c}{\rho_m}$ is about 3.

Summary.—The radius of a spherical aggregate of cold or degenerate matter of mass M is determined, full allowance being made for the variation of atomic weight through the configuration which results from the theory of pressure ionization. The resulting relations between M and R , the radius, are shown graphically in fig. 1, and are closely similar to those obtained by Kothari on the simplifying assumption of uniform atomic weight.

It is a pleasure to record my grateful thanks to Dr. D. S. Kothari for his kind interest in this work.

Appendix

TABLE I (a)

$z_0 = 1000, x_1 = .0818, v_1 = 3.01 \times 10^4$

x	z	v
0	1000	0
.01	979	310
.02	931	2320
.03	855	6790
.04	756	1.32×10^4
.05	638	2.00×10^4
.06	505	2.57×10^4
.07	351	2.91×10^4
.08	130	3.01×10^4
.0818	1.25	3.01×10^4

TABLE I (b)

$z_0 = 500, x_1 = .1154, v_1 = 1.07 \times 10^4$

x	z	v
0	500	0
.01	495	40.1
.02	482	311
.03	463	976
.04	436	2.09×10^3
.05	403	3.58×10^3
.06	365	5.29×10^3
.07	322	7.00×10^3
.08	276	8.51×10^3
.09	225	9.65×10^3
.10	168	1.04×10^4
.11	95.5	1.06×10^4
.1154	1.25	1.07×10^4

TABLE I (c)

$$z_0 = 100, x_1 = .2545, v_1 = 937$$

x	z	v	x	z	v
0	100	0	.14	70.6	514
.01	99.8	.331	.15	66.8	582
.02	99.3	2.63	.16	62.8	647
.03	98.5	8.75	.17	58.6	708
.04	97.3	20.3	.18	54.2	763
.05	95.8	38.6	.19	49.7	811
.06	94.0	64.5	.20	45.0	851
.07	92.0	98.6	.21	40.0	884
.08	89.6	141	.22	34.7	908
.09	87.0	190	.23	28.8	924
.10	84.2	247	.24	21.9	934
.11	81.1	309	.25	12.5	937
.12	77.8	376	.2545	1.25	937
.13	74.3	444			

TABLE I (d)

$$z_0 = 10, x_1 = .6939, v_1 = 23.6$$

x	z	v
0	10	0
.1	9.77	.307
.2	9.23	2.28
.3	8.38	6.57
.4	7.27	12.4
.5	5.91	18.2
.6	4.24	22.3
.6939	1.25	23.6

TABLE I (e)

$$z_0 = 4, x_1 = .840, v_1 = 3.84$$

x	z	v
0	4	0
.1	3.95	.020
.2	3.85	.158
.3	3.68	.494
.4	3.44	1.05
.5	3.14	1.77
.6	2.78	2.55
.7	2.33	3.26
.8	1.72	3.75
.84	1.25	3.84

TABLE I (f)

$$z_0 = 3, x_1 = .8239, v_1 = 1.84$$

x	z	v
0	3	0
.1	2.97	.009
.2	2.90	.067
.3	2.79	.213
.4	2.64	.460
.5	2.44	.792
.6	2.20	1.17
.7	1.89	1.53
.8	1.44	1.80
.8239	1.25	1.84

TABLE I (g)

$$z_0 = 2, x_1 = .6838, v_1 = .449$$

x	z	v
0	2	0
.05	1.996	3.31×10^{-4}
.1	1.986	2.63×10^{-3}
.2	1.94	2.02×10^{-2}
.3	1.88	6.43×10^{-2}
.4	1.78	1.40×10^{-1}
.5	1.65	.243
.6	1.48	.360
.6838	1.25	.449

TABLE I (*h*)

$z_0 = 1.5, x_1 = .4293, v_1 = 6.59 \times 10^{-2}$

x	z	v
0	1.5	0
.1	1.48	1.09×10^{-3}
.2	1.44	8.35×10^{-3}
.3	1.38	2.62×10^{-2}
.4	1.29	5.56×10^{-2}
.4293	1.25	6.59×10^{-2}

TABLE II

z_0	x_1	v_1
1000	.0818	3.01×10^4
500	.1154	1.07×10^4
100	.2545	937
10	.6939	23.6
4	.840	3.84
3	.8239	1.84
2	.6838	.449
1.5	.4293	6.59×10^{-2}
1.35	.2792	1.56×10^{-2}
1.3	.1971	5.23×10^{-3}
1.26	.0878	4.44×10^{-4}
1.255	.0620	1.56×10^{-4}
1.251	.0277	1.39×10^{-5}

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THE DEPENDENCE OF THE SOLAR VELOCITY ON THE DISTANCE FROM THE PLANE OF THE MILKY WAY.

P. P. Bruna.

(Communicated by Dr. P. J. van Rhijn)

Introduction

P. v. d. Kamp and A. R. Vyssotsky* have pointed out that $p_\tau < p_v$ for galactic stars, whereas $p_\tau > p_v$ for stars at high galactic latitudes. p_τ is the parallax derived from components of proper motion perpendicular to the direction of the Sun's motion (τ -components), and p_v is the parallax derived from components of proper motion in the direction of the ant-apex of the solar motion (v -components). They base this conclusion on the proper motions of *McCormick Publications*, Vol. VII. Table I is computed on the basis of the data of Tables 6.10 and 6.12 of the publication quoted.†

TABLE I

Gal. Lat.	$\frac{p_\tau}{p_v}$				
	Mag. > 12.5	11.5-12.4	10.5-11.4	9.5-10.4	7.5-9.4
$\pm 41^\circ \pm 90^\circ$	1.53	1.06	1.35	1.13	1.25
$\pm 21^\circ \pm 40^\circ$	1.73	1.56	1.00	1.39	1.29
$\pm 11^\circ \pm 20^\circ$	1.50	1.05	1.13	1.20	1.00
$0^\circ \pm 10^\circ$	.84	.89	.90	.82	.86

p_τ and p_v have been computed by the formulæ

$$p_\tau = \frac{4.74}{V_\tau} \times \bar{\tau}; \quad p_v = \frac{4.74}{V_s} \times \left(\frac{\bar{h}}{\rho} \right), \quad (I)$$

where V_τ = arithmetical mean value in km./sec. of the linear velocity in the τ -direction, derived from the ellipsoidal velocity distribution; $\bar{\tau}$ = arithmetical mean value in seconds of arc per year of the τ -components of the proper motions. It has been corrected for the systematic effect caused by the accidental errors in the proper motions. V_s = velocity of the Sun in km./sec. $(\bar{h}/\rho)$ = secular parallax. From (I) we derive

* *Publications McCormick Observatory*, 7, 34, 1937.

† This table does not agree with Table I of E. Williams' publication (*Ap. J.*, 84, 346, 1936), because that table, as Mrs. Williams-Vyssotsky communicated to me, was based on preliminary values of p_τ , which differ from the final values.

$$p_{\tau}/p_v = \frac{V_s}{V_{\tau}} \frac{\bar{\tau}}{(\bar{h}/\rho)}. \quad (\text{II})$$

V_{τ} and V_s have both been taken independent of the galactic latitude in the computation of Table I. Assuming that $\bar{\tau}$ and $(\bar{h}/\rho)$ are free from systematic errors, the increase of $\frac{p_{\tau}}{p_v}$ with increasing arithmetical value of the galactic latitude must be ascribed either to the V_s used by Miss Williams at high latitude being too large, or the V_{τ} being too small. Either the true value of V_s , therefore, must decrease or the true value of V_{τ} must increase with increasing distance from the galactic plane.

Emma T. R. Williams* is of opinion that the former supposition holds true. She bases her conclusion on the data compiled in Table II. z means the distance from the galactic plane in parsecs. ΔV_s is the difference in V_s between groups with large and small distance from the galactic plane.

TABLE II

Remarks	V_s		V_s		ΔV_s	Class of Stars	Material
	parsecs	km./sec.	parsecs	km./sec.	km./sec.		
<i>a</i>	28	21.4	64	13.1	-8.3	A	Rad. vel.
<i>b</i>	33	19.0	67	11.5	-7.5	G-giants	"
<i>c</i>	26	27.0	70	21.6	-5.4	K	"
<i>d</i>	25	21.9	60	17.7	-4.2	B-M	Rad. vel. + p.m.
<i>e</i>	17	15.6	60	12.9	-2.7	B8-A5	" "
<i>f</i>	19	19.5	56	14.0	-5.5	B-M	Proper motions.
<i>g</i>	15	20.6	110	15.7	-4.9	Giants	Space motions.
<i>d-g</i>	19	19.4	72	15.1	-4.3		

Remarks on the Table

a, b, c. W. Gyllenberg and K. G. Malmquist.† F stars have been omitted on account of their small difference in z . No correction has been applied for differential galactic rotation.

d. Gyllenberg and Malmquist.† V_s together with the co-ordinates of the apex and the mean distance is solved by the method of least squares from proper motions and radial velocities together. No correction has been applied for differential galactic rotation.

e. R. A. Robb.‡ Method as *d.* Parallax stars. Only the B8 and B9 stars have been corrected for differential galactic rotation.

f. R. D. H. Jones.§ V_s has been derived from proper motions

* *Ap. J.*, **84**, 346, 1936.

† *Lund Meddelanden*, **1**, No. 108, 1925.

‡ *M.N.*, **94**, 406, 1934.

§ *M.N.*, **91**, 563, 1931.

exclusively in an investigation of star-streams. The unit of V_s is $\frac{1}{h}$, h being the modulus of precision of the Gaussian distribution of the components of peculiar velocity. I have, more or less arbitrarily, assumed h so as to make V_s in the lowest galactic latitude group equal to 19.5 km./sec. No correction has been applied for differential galactic rotation.

g. V_s has been derived by E. Williams* from space velocities of giants, published by Balanovsky.† Miss Williams does not state whether she has applied a correction for differential galactic rotation.

The proper motions in all these investigations have been taken from Boss's P.G.C. In the space velocities of Balanovsky these have been supplemented by proper motions from Cincinnati. The results of Table II are therefore not wholly independent, the material of the investigations being partially the same.

Table II shows that in all investigations V_s is smaller at some distance from the galactic plane than near the galaxy. The material of the investigations *a*, *b* and *c* is also used in *d*. The values of the former three investigations are therefore omitted from the final values in the last line (*d-g*), which are the simple mean values of the results of *d*, *e*, *f* and *g*.

An investigation by H. Nordström of radial velocities shows a much smaller decrease of V_s with increasing distance from the galactic plane.‡ E. Williams is of opinion that this result must be ascribed to the exclusion of stars with large radial velocity. I have tested this hypothesis in a note at the end of the present article.

Sun's Velocity from B and c Stars

I have re-examined the dependence of the solar velocity from the distance to the galactic plane by means of radial velocities of B and *c* stars. A relatively small number of these stars gives a rather accurate value of the Sun's velocity, the peculiar velocities being small. A disadvantage is the strong galactic concentration. On the other hand, the distances are large on account of the high luminosity; therefore stars in relatively small galactic latitudes have a large distance from the galactic plane.

I. O and B Stars.—I have used all stars which occur in a list by Stebbins and Huffer§ and which have a constant radial velocity according to Plaskett and Pearce.|| The radial velocities have been taken from the latter source exclusively.

Stebbins and Huffer have determined distances of these stars by the formula $5 \log r = m - M + 5 - 2E$, where r = distance in parsecs, m = apparent visual magnitude, M = absolute visual magnitude, ¶ E = photoelectric colour excess, $2E$ is the adopted visual absorption.

* *Ap. J.*, **84**, 346, 1936.

† *Bull. Astr. Inst. Leningrad*, No. 11, 1925.

‡ *Lund Meddelanden*, **2**, No. 79, Table 33, 1936.

§ *Washburn Publ.*, **15**, 5, 1934.

|| *Victoria Publ.*, **5**, 2, 1931.

¶ *Loc. cit.*, Table IX.

The distances given by Stebbins and Huffer have been checked by Oort by means of differential galactic rotation terms, derived from radial velocities. Oort * gives the mean distances for certain groups of stars between the limits of galactic latitudes 0° to 2° , 3° to 5° , 6° to 11° and $\geq 12^\circ$ computed from the differential galactic rotation. They have been reprinted in the third column of Table III; the second column gives the mean distances according to Stebbins and Huffer † for the same stars.

TABLE III

 $\bar{r}$ in parsecs

b	St. and H.	Oort
$0^\circ - 2^\circ$	814	780
$3^\circ - 5^\circ$	684	790
$6^\circ - 11^\circ$	606	940
$\geq 12^\circ$	547	860

The two sets of distances agree substantially in low galactic latitudes, but disagree seriously in high latitudes. Oort attributes the discrepancy to the visual absorption being taken too small, and the absolute magnitude too large. If we adopt a visual absorption of $4E$ instead of $2E$ and apply a correction of -0.82 to the absolute magnitudes, the mean distances are in fairly close accordance with the values of Oort for all latitudes.

Because in Oort's Table 4 only Bo-B2 stars are used, and our B stars are for the greater part of later spectral type (see Table IV), we must investigate whether the other types require the same correction in the absolute magnitudes. If we compute from Oort's Table 5 the absolute magnitude M with the formula

$$M = m + 5 - 5 \log r - 4E,$$

where m = apparent magnitude, r = distance in parsecs, E = average colour excess, we find for the mean of the two Bo-B2 groups, of the four B3-B4 groups, and of the four B5-B9 groups $M = -3.6$, -1.4 , -1.2 respectively. Stebbins and Huffer assumed for these spectral groups -2.5 , -1.1 and -0.3 . † According to Oort, the latter values are therefore too faint by an amount of $1^m.1$, $0^m.3$ and $0^m.9$ respectively; we have applied a mean correction of -0.8 to the absolute magnitudes of all B stars.

The equation of condition for the solution of the motion of the Sun, galactic rotation and K -term is

$$V = K - X_s \cos l \cos b - Y_s \sin l \cos b - Z_s \sin b + Ar \sin z(l - l_0) \cos^2 b. \quad (\text{III})$$

In this formula l = galactic longitude, b = galactic latitude of the star, l_0 = galactic longitude of the centre of rotation, assumed as 324° , r = distance from the Sun in parsecs, V = observed radial velocity, K = constant, X_s , Y_s , Z_s are the components of the Sun's velocity, X -axis in the galaxy towards longitude 0° , Y -axis in the galaxy towards longitude 90° , Z -axis towards North galactic pole. Adopted pole of the Milky Way, R.A. $12^h 44^m.4$, Dec. $+26^\circ.8$ (1900).

* *B.A.N.*, 8, 233, Table 4, 1938.† *Loc. cit.*, p. 242.‡ *Washburn Publ.*, 15, 5, Table 9, 1934.

The stars used in this investigation having a strong galactic concentration, it is impossible to make an accurate determination of Z_s . This is of minor importance for the total velocity of the Sun, because Z_s is rather small. We take $Z_s = +6.84$ km./sec., derived from $V_s = +20$ km./sec. and a galactic latitude of the solar apex $B = +20^\circ$. If the probable errors of the assumed values of V_s and B are ± 2 km./sec. and $\pm 5^\circ$ respectively, the probable error of $Z_s = \pm 1.8$ km./sec. and the probable error of V_s , due to the uncertainty of Z_s , is ± 0.6 km./sec.

A = Oort's coefficient of the differential galactic rotation, which we take equal to 0.0155 km./sec. parsec in accordance with Plaskett and Pearce.* The equation of condition becomes then

$$V' = K - X_s \cos l \cos b - Y_s \sin l \cos b, \quad (\text{IV})$$

where V' is defined by

$$V' = V + 6.84 \sin b - 0.0155r \sin 2(l - l_0) \cos^2 b.$$

The stars have been divided into three groups according to the value of $z = r \sin b$, where r = distance derived by Stebbins and Huffer. The distribution of the stars according to the distance from the galactic plane z in parsecs, and the spectral type, is shown in Table IV. The distribution according to spectral type is nearly the same for the three groups.

TABLE IV

	$Z=0-35$		70-99		≥ 100	
	Number	Per cent.	Number	Per cent.	Number	Per cent.
Od	1	$\frac{1}{2}$				
Oe	2	1	1	1	1	$1\frac{1}{2}$
Oe5	2	1				
Op	1	$\frac{1}{2}$				
Bo	22	11	7	8	8	12
B1	5	$2\frac{1}{2}$	3	$3\frac{1}{2}$	5	7
B2	29	14	11	$12\frac{1}{2}$	5	7
B3	67	33	36	41	35	$51\frac{1}{2}$
B5	72	35	29	33	14	21
B	2	1	1	1		
B8	1	$\frac{1}{2}$				

The values of K , X_s and Y_s have been derived for each group by the method of least squares. The results are in Table V. $\bar{z}$ is the mean distance from the galactic plane, corrected according to p. 253.

II. *c Stars*.—I have used stars with constant radial velocity which, according to the Henry Draper Catalogue, have narrow or very narrow lines. The radial velocities have been taken from Moore's Catalogue.† I have made a solution for galactic latitude $< 10^\circ$ and for galactic latitude $> 10^\circ$. The equation of condition is the same as in the case of the B stars

* *M.N.*, 94, 679, 1934.

† *Lick Publ.*, 18.

TABLE V

$ z $	Number	V_s	p.e. of V_s	X_s	Y_s	Z_s *	K	p.e. of K	Mean Residual of a Single Star
parsecs		km./sec.	km./sec.	km./sec.	km./sec.	km./sec.	km./sec.	km./sec.	km./sec.
23	204	20.5	± 1.8	+17.9	+ 7.2	+6.8	+0.1	± 2.0	± 9.5
117	88	19.9	± 1.5	+18.7	- 0.2	+6.8	+0.2	± 1.8	± 7.6
215	68	25.4	± 2.7	+21.0	+12.6	+6.8	+0.3	± 2.9	± 12.8

* Assumed.

formula (IV)). We take, in accordance with a publication by Oort,* $r = \frac{1.15}{\bar{p}}$, where $\bar{p}$ is the mean parallax. It has been interpolated from Oort's * Table 3.

The result of the least squares solution is in Table VI.

TABLE VI

$ b $	$ z $	Number	V_s	p.e. of V_s	X_s	Y_s	Z_s *	K	p.e. of K	Mean Residual of a Single Star
	parsecs		km./sec.		km./sec.]	km./sec.		km./sec.		km./sec.
< 10°	34	68	20.4	± 1.9	+16.4	+10.0	+6.8	-6.4	± 1.2	11.5
> 10°	157	25	17.3	± 3.0	+12.5	+ 9.8	+6.8	+1.6	± 2.0	11.2

* Assumed.

The mean value of z , the distance from the galactic plane, is found as follows:—

In the group with latitude $> 10^\circ$ it is not permitted to compute the mean z by means of the formula $z = \bar{r} \sin b$ because a correlation exists between r and b on account of the strong galactic concentration of the stars considered. I have made an estimate of the value of $r \sin b - \bar{r} \sin b$ for this group by assuming: (1) That the surfaces of equal density are planes parallel to the Milky Way. (2) That the decrease of the density d with increasing distance from the galactic plane z is given by the formula $d = e^{-az^2}$. (3) That the mean absolute magnitude of the c stars is -3.2 and the dispersion of the absolute magnitudes is 0.8 mag. (Schilt, *B.A.N.*, 2, 50, 1924). Assuming a value of a and taking into account the interstellar absorption, we compute the mean value of z for stars between specified intervals of magnitude and galactic latitude, and finally find $\bar{r} \sin b$ and $r \sin b$ for all

* *Bull. Astr. Inst. Netherlands*, 4, 82, 1927.

stars combined. We have made this computation with two values of a for which the corresponding values of $\bar{r} \sin b$ are larger and smaller than the observed value respectively. The values of $r \sin b - \bar{r} \sin b$ are in the two cases nearly the same (24 and 26 parsecs respectively).

The results given in Table VI will presently be combined with the values of Table V. But we must first of all investigate whether the difference in solar motion for the groups of Table VI may be wholly or partly due to a difference in spectral composition of the two groups. This distribution is shown in Table VII.

TABLE VII
Distribution of c Stars according to Spectral Type

Spectral Type	$b < 10^\circ$		$b > 10^\circ$		V_s^*
	Number	Per cent.	Number	Per cent.	
Bo-B9	26	38	5	20	23.0
Ao-A9	17	25	3	12	17.6
Fo-F9	16	24	11	44	19.9
Go-G9	8	12	6	24	21.5
Ko-K9	1	1	...	...	22.5

The frequency of the B and A stars in the low galactic latitude group is twice the value in the high latitude group. For the F and G stars the reverse holds true. The influence of this difference in spectral composition on the computed value of V_s is probably small. The value of the solar motion is given in the last column of Table VII.* Its mean value, using weights proportional to the numbers of stars, is 20.7 km./sec. for the low latitude group and 20.6 km./sec. for the high latitude group.

The influence of the difference in spectral composition on the difference in the solar motion of the groups may be eliminated by giving each B and A star in the low latitude group a weight 1/4. The solar velocity then becomes 19.4 instead of 20.4 km./sec., so that the difference between the value at high and low galactic latitudes is decreased by 1 km./sec.

III. *Combination of the Results in Table VIII.*—In the left part of the table we have collected the results of Table V and Table VI. The first two values have been combined into a single mean, and similarly the last two values. They are in the right-hand part of the table.

TABLE VIII

	$ \bar{z} $	V_s	p.e. of V_s	$ \bar{z} $	V_s	p.e. of V_s
	parsecs	km./sec.	km./sec.	parsecs	km./sec.	km./sec.
B stars	23	20.5	± 1.8			
c „	34	20.4	2.1	28	20.5	1.4
B „	117	19.9	1.5	117	19.9	1.5
c „	157	17.3	3.0	189	21.8	2.0
B „	215	25.4	2.7			

* *Zeitschrift für Astrophysik*, 10, No. 3, Tabelle 9, 1935.

IV. *Conclusion.*—According to the present investigation there is no decrease of V_s with increasing distance from the Milky Way over a distance of 161 parsecs, whereas according to Table II we find a decrease of 4 km./sec. over a distance of 53 parsecs. The former result, extending over a larger interval of distance, seems more trustworthy.

Note 1.—As is stated on p. 250, Nordström finds a small decrease of V_s with increasing z , and Miss Williams is of opinion that a much higher decrease would have been found if stars with large radial velocities had not been excluded. Nordström has rejected all stars for which the radial velocity deviates more than three times the dispersion σ from the mean value valid for a particular square. The mean value of σ for all squares equals 17.8 km./sec.* Now it is well known that stars of large peculiar space velocity give a larger value of the solar motion than other stars. They move, moreover, by preference parallel to the galactic plane, so that Nordström by his procedure excludes more stars with large peculiar radial velocity in the Milky Way than at high latitudes. By excluding the stars of high velocity the solar motion in both the low latitude and the high latitude groups will be decreased, but the decrease will be larger for the low latitude group. Hence the decrease of the solar velocity with increasing distance from the galactic plane will be reduced.

This supposition of Miss Williams may be tested by a calculation by Oort, who has computed the influence of the exclusion of peculiar radial velocities > 62 km./sec. on the solar motion of Gyllenberg and Malmquist.† The results are in Table IX.

TABLE IX

Galactic Latitude	Stars of High Radial Velocity			
	Included		Excluded	
	V_s	Number	V_s	Number
$0^\circ \pm 30^\circ$	22.9	1334	21.8	1243
$\pm 30^\circ \pm 90^\circ$	18.2	955	16.9	848
	4.7		4.9	

The last line contains the difference in solar motion between high and low galactic latitudes. The Sun's velocity is indeed smaller after the stars with large peculiar radial velocity have been excluded. The difference, however, is nearly the same in both latitude groups; the difference in solar motion between high and low galactic latitudes, therefore, is not changed by the exclusion of the high velocity stars.

* *Loc. cit.*, Table 36.

† *Groningen Publ.*, 40, Table 22, 1926.

We have also examined the effect of the high velocity stars on V_s by means of the data of *Lick Publication*, 16, Table V. Mean radial velocities for different squares are tabulated in that table, stars of peculiar radial velocity exceeding 60 km./sec. having been excluded. The excluded stars are in Table IV of the publication quoted. We have first computed V_s separately for squares with mean galactic latitude $< 30^\circ$ and $> 30^\circ$, excluding the stars of high velocity; next the high velocity stars with visual magnitude < 5.51 were included. Results in Table X.

TABLE X

Galactic Latitude	Stars of High Radial Velocity			
	Included		Excluded	
$0^\circ \pm 30^\circ$	V_s 21.7	Number 1361	V_s 19.6	Number 1335
$\pm 30^\circ \pm 90^\circ$	17.5	701	18.3	691
	4.2		1.3	

The last line contains again the difference in V_s between high and low galactic latitudes. The influence of the high velocities is large in low galactic latitudes; in high latitudes its sign is reversed. By including the stars of high velocity, therefore, the solar velocity in low latitudes becomes appreciably larger than the value at high latitudes.

As the results of Table IX and Table X do not agree, no definite conclusion can be drawn from it.

Note 2.—P. J. van Rhijn has pointed out* that part of the difference between p_τ and p_v , as found by v. d. Kamp and Vyssotsky, may be explained by the fact that they have assumed, in accordance with Wilson and Raymond,† a very large value of the Sun's velocity for F–M dwarfs, viz. 34 km./sec. for spectrum F8–G5 and 35 km./sec. for K0–Md. If this value is too high, the mean solar velocity in high latitudes is found too large relatively to the value in low latitudes, because the percentage of dwarf stars is taken larger in high than in low latitudes. Contrary to the statement in *Publications McCormick Observatory*, 7, 32, v. d. Kamp and Vyssotsky have derived the relative numbers of giants and dwarfs from Table 4.3, excluding the results of their own material (communicated by a private letter).

We will compute the Sun's velocity V_s and p_τ/p_v , adopting 22.2 km./sec. for the Sun's velocity of the dwarf stars.‡

In Table XI we have compiled the values of the Sun's velocity used by v. d. Kamp and Vyssotsky. V_g = Sun's velocity for giants, V_d = Sun's velocity for dwarfs.

* *Hemel en Dampkring*, 1938 January.

† *Astronomical Journal*, 40, 125, 1930.

‡ *Groningen Publ.*, 37, 13, 1925.

TABLE XI

Spectrum	V_g	V_d
F8-G5	16.0	34.0
K0-K2	17.9	35.0
K5-M	18.4	35.0

In *Pub. McCormick Obs.*, 7, Table 6.10, we find the assumed values of the solar velocity for giants and dwarfs together and in different galactic latitude groups. If we suppose that a percentage e are dwarfs we can put

$$V_s = \frac{eV_d + (1-e)V_g}{100}. \quad (\text{V})$$

The values of e , computed from formula (V), are tabulated in Table XII.

TABLE XII

Assumed Percentages of Dwarf Stars

Spectrum Gal. Lat.	F5-G0	G5-G7	K0-K2	K5-M
0°-10°	44	44	6	4
11°-20°	44	44	12	4
21°-40°	50	50	24	4
41°-90°	56	56	42	10

Taking for the Sun's velocity relatively to the dwarfs (V_d') 22.2 km./sec. instead of the values of Table XI, we have the relation

$$V_s' = \frac{eV_d' + (1-e)V_g}{100}, \quad (\text{VI})$$

where V_s' is the Sun's velocity relative to the mixture of giants and dwarfs. Adopting e in accordance with Table XII, we find V_s' by the formula (VI) for groups of specified spectral class and galactic latitude. We have further taken mean values of V_s and V_s' for all spectral classes, the weights being proportional to the numbers of *loc. cit.*, Table 6.1. The results for V_s' are in Table XII. Adopting for the solar velocity V_s' instead of V_s , we find the values p_τ/p_v' in the sixth column of Table XIII for the ratio of the parallaxes derived from the τ and v components of proper motion. The values in the fifth column are reprinted from Table I for the magnitude 9.5-10.4, which is the mean magnitude of the stars used by v. d. Kamp and Vyssotsky. They represent the McCormick ratio p_τ/p_v .

TABLE XIII

Gal. Lat.	V_s	V_s'	V_s'/V_s	p_τ/p_v	p_τ/p_v'
$\pm 0^\circ \pm 10^\circ$	18.6	17.2	0.92	0.82	0.75
$\pm 11^\circ \pm 20^\circ$	19.8	17.8	0.90	1.20	1.08
$\pm 21^\circ \pm 40^\circ$	21.9	18.5	0.84	1.39	1.17
$\pm 41^\circ \pm 90^\circ$	24.3	19.2	0.79	1.13	0.89

It appears from the table that the increase of p_τ/p_v with increasing galactic latitude disappears, if we reduce the Sun's velocity for dwarf stars to the value found in *Groningen Publications*, 37.

I wish to thank Dr. Vyssotsky and Mrs. Williams-Vyssotsky for corrections to the quoted publications, and Professor van Rhijn for his assistance during the preparation of this paper.

Summary

P. v. d. Kamp and A. R. Vyssotsky have found that the ratio of the mean parallaxes $\frac{p_\tau}{p_v}$, derived from τ and v components of proper motion with a constant solar velocity, increases from the Milky Way to the galactic poles. The v -component is the component of proper motion in the direction of the solar ant-apex, the τ is the component perpendicular to this direction.

Emma T. R. Williams thinks that this is due to the solar motion decreasing with increasing distance from the galactic plane.

The arguments of E. Williams in favour of this supposition are discussed in the Introduction and Note 1.

In sections I, II, III and IV the dependence of the solar velocity on the distance from the galactic plane is re-examined by means of radial velocities of O, B and c stars. The final results are in Table VIII. They do not show the decrease of the solar motion with increasing distance from the galactic plane expected by E. Williams.

In Note 2 another explanation is given of the increase of p_τ/p_v with increasing galactic latitude.

TABLES OF THE TWO CHIEF LAPLACE COEFFICIENTS.

Alan Fletcher.

1. The quantities tabulated in this paper are the coefficients b_0 and b_1 which occur in the expansion

$$(1 + \alpha^2 - 2\alpha \cos \phi)^{-1/2} = \frac{1}{2}b_0 + \sum_1^{\infty} b_n \cos n\phi,$$

where α usually denotes the ratio of the mean distances of two planets from the Sun. Various notations are in use for the coefficients. For normal values of α the chief existing tables, those of Runkle * and those of Brown and Brouwer,† are logarithmic. Some corrections to Runkle's values have been given by Witt.‡

It is known that b_0 and b_1 are given by

$$b_0 = \frac{4}{\pi}K, \quad b_1 = \frac{4}{\pi} \frac{K - E}{\alpha},$$

where K and E are the complete elliptic integrals with modulus (the usual k) equal to α . These integrals are almost invariably tabulated with argument either $\sin^{-1} k$ (after Legendre) or, more recently, k^2 . Both of these arguments are convenient for general mathematical purposes on account of the ease with which functions of the complementary modulus may be dealt with. Convenient provision for interpolation is, however, lacking, and so also are tables with k as argument. As a by-product of the present work, tables of K and E with argument k have been constructed, and it is hoped to publish these elsewhere. If a number of Laplace coefficients are to be tabulated, the argument k ($\equiv \alpha$) is very much to be preferred.

2. All the values tabulated below were computed independently of existing tables, and independently for each value of α . For every value of α considered the arithmetic-geometric mean was computed to 12 or more decimal places by the usual formulæ

$$\begin{aligned} A_0 &= 1, & B_0 &= \sqrt{1 - \alpha^2}, \\ A_n &= \frac{1}{2}(A_{n-1} + B_{n-1}), & B_n &= \sqrt{A_{n-1}B_{n-1}}, \end{aligned}$$

where the quantities A_n and B_n quickly converge to a limit M . Then b_0 was determined to 11 or more decimal places from

$$b_0 = 2/M.$$

* *Smithsonian Contributions*, 1855.

† *Trans. Yale Univ. Obs.*, 6 (5), 1932.

‡ *Astr. Nachr.*, 246, 221, 1932.

When b_1 was to be found, the quantity S was computed from

$$S = \frac{1}{2}\alpha^2 + D_1 + D_2^2 + 2D_3^2 + 4D_4^2 + 8D_5^2 + \dots,$$

where $D_n = A_{n-1} - B_{n-1}$. This formula was found to be more convenient and accurate for numerical purposes than various equivalent forms. Then b_1 was determined to 11 or more decimal places from

$$b_1 = S/M\alpha.$$

It is believed, on the strength of differencing, that the 11-decimal values of b_0 and b_1 so deduced rarely differ from the true values by more than one unit of the last place, and that errors greater than two such units are entirely absent. Some qualification of this estimate is perhaps necessary for very high values of α , for which only $b_0 - b_1$ can be effectively differenced with the interval used, but all values of M for $\alpha > .93$ have been checked by duplicate computations, most of them kindly performed by Dr. J. C. P. Miller.

The main Table I gives exact 8-decimal values of b_0 and b_1 , with modified second differences, together with the digits in the 8th to 11th decimal places of each coefficient, for convenience in case of further tabulation. Thus 11-decimal values may usually be obtained by placing the digits given in the columns headed "8-11" after the 8-decimal values shorn of the last digit. When the digits 8-11 are greater than 9500, they are printed in italics, as a reminder to change the 7th and possibly earlier decimal in writing down 11-decimal values.

Since values of α higher than .75 occur infrequently in practice, Table I is that which will normally be used, and pains have been taken to make the 11-decimal values reasonably correct. The processes used were sufficient to give some indication of the 12th decimal, and all rather high differences of the 11-decimal values were examined in relation to the rounding-off to 11 places. A considerable amount of inspection and recomputation failed to disclose any errors of more than one final unit, although the existence of one or two errors rather larger than a unit is perhaps not impossible.

The modification of the differences of the 8-decimal values is that introduced by Comrie,* and described by him as the "throw-back." The quantities tabulated are thus, in the usual central-difference notation,

$$\delta_m^2 = \delta^2 - 0.184\delta^4.$$

Throughout this table δ^4 is less than 1000, the usual limit for the employment of the above simple throw-back, so that interpolation may be performed by means of Everett's formula with second differences. When suitable tables of the Everett coefficients are not available, the interpolation may be performed directly from the formula

$$f_\theta = \phi f_0 + \theta f_1 - \frac{1}{6}\theta\phi \{ (1 + \phi)\delta_m^2 f_0 + (1 + \theta)\delta_m^2 f_1 \},$$

where $\phi = 1 - \theta$.

* *M.N.*, 88, 506, 1928; *Interpolation and Allied Tables* (reprinted from the *Nautical Almanac* for 1937), pp. 787 and 928.

TABLE I

a	b_0	δ_m^2	8-II	b_1	δ_m^2	8-II
·00	2·00000 000	+ 9 998	000 0	·00000 000	+ 0	000 0
·01	2·00005 000	10 004	028 1	·01000 038	223	750 2
·02	2·00020 005	10 011	450 1	·02000 300	451	007 5
·03	2·00045 023	10 030	279 5	·03001 013	675	307 0
·04	2·00080 072	10 054	208 0	·04002 402	904	240 3
·05	2·00125 176	+10 083	608 7	·05004 695	+ 1 129	483 8
·06	2·00180 365	10 123	541 4	·06008 118	1 361	827 3
·07	2·00245 678	10 164	758 8	·07012 902	1 590	203 3
·08	2·00321 157	10 219	714 5	·08019 277	1 825	716 0
·09	2·00406 856	10 275	572 6	·09027 477	2 058	671 9
·10	2·00502 832	+10 344	218 2	·10037 736	+ 2 298	609 8
·11	2·00609 153	10 415	270 6	·11050 293	2 538	332 6
·12	2·00725 891	10 498	097 1	·12065 389	2 784	939 4
·13	2·00853 128	10 588	829 2	·13083 269	3 029	858 6
·14	2·00990 954	10 683	380 4	·14104 179	3 282	882 3
·15	2·01139 465	+10 788	465 8	·15128 372	+ 3 538	201 2
·16	2·01298 766	10 904	623 9	·16156 104	3 800	442 0
·17	2·01468 972	11 027	240 0	·17187 637	4 065	705 1
·18	2·01650 206	11 156	572 5	·18223 236	4 337	604 5
·19	2·01842 598	11 299	780 9	·19263 173	4 615	309 5
·20	2·02046 290	+11 448	956 5	·20307 726	+ 4 899	587 6
·21	2·02261 432	11 608	155 8	·21357 179	5 188	850 9
·22	2·02488 184	11 782	436 4	·22411 822	5 489	203 1
·23	2·02726 719	11 960	896 2	·23471 955	5 795	490 7
·24	2·02977 217	12 155	715 3	·24537 884	6 108	355 2
·25	2·03239 872	+12 358	201 9	·25609 923	+ 6 433	290 0
·26	2·03514 888	12 577	841 3	·26688 397	6 767	699 5
·27	2·03802 483	12 807	349 1	·27773 640	7 110	961 7
·28	2·04102 887	13 049	728 2	·28865 995	7 466	495 6
·29	2·04416 343	13 307	331 4	·29965 818	7 834	831 6
·30	2·04743 109	+13 580	927 5	·31073 477	+ 8 211	687 6
·31	2·05083 458	13 867	773 7	·32189 350	8 608	049 6
·32	2·05437 677	14 173	693 4	·33313 833	9 011	257 7
·33	2·05806 072	14 493	160 4	·34447 331	9 437	098 5
·34	2·06188 964	14 834	389 8	·35590 269	9 877	903 7
·35	2·06586 694	+15 196	436 6	·36743 087	+10 332	656 5
·36	2·06999 623	15 573	301 8	·37906 241	10 811	104 9
·37	2·07428 130	15 978	048 0	·39080 209	11 306	885 1
·38	2·07872 619	16 402	924 7	·40265 487	11 822	652 9
·39	2·08333 515	16 852	503 4	·41462 592	12 366	226 7
·40	2·08811 268	+17 330	825 8	·42672 067	+12 931	741 5

TABLE I—continued

α	b_0	δ_m^2	8-11	b_1	δ_m^2	8-11
.40	2.08811 268	+17 330	825 8	.42672 067	+12 931	741 5
.41	2.09306 356	17 832	563 3	.43894 478	13 522	816 1
.42	2.09819 282	18 368	192 0	.45130 417	14 144	733 9
.43	2.10350 582	18 933	182 6	.46380 506	14 797	639 4
.44	2.10900 822	19 536	208 1	.47645 398	15 479	751 8
.45	2.11470 604	+20 170	370 0	.48925 776	+16 203	598 8
.46	2.12060 564	20 851	447 0	.50222 363	16 959	269 5
.47	2.12671 382	21 566	165 8	.51535 917	17 760	693 6
.48	2.13303 775	22 333	500 2	.52867 239	18 607	944 5
.49	2.13958 510	23 146	998 2	.54217 176	19 497	574 2
.50	2.14636 401	+24 016	143 0	.55586 620	+20 444	979 3
.51	2.15338 318	24 936	750 7	.56976 518	21 447	805 6
.52	2.16065 184	25 928	410 1	.58387 874	22 511	393 7
.53	2.16817 990	26 981	969 0	.59821 753	23 641	271 6
.54	2.17597 791	28 111	074 2	.61279 287	24 848	701 1
.55	2.18405 718	+29 321	771 0	.62761 683	+26 130	283 9
.56	2.19242 982	30 618	171 3	.64270 226	27 505	636 2
.57	2.20110 882	32 012	199 9	.65806 291	28 973	141 4
.58	2.21010 814	33 513	429 0	.67371 348	30 544	792 1
.59	2.21944 280	35 127	014 4	.68966 971	32 236	133 4
.60	2.22912 897	+36 874	749 7	.70594 853	+34 052	323 7
.61	2.23918 413	38 754	253 5	.72256 813	36 011	331 3
.62	2.24962 713	40 799	314 6	.73954 813	38 124	285 8
.63	2.26047 844	43 014	416 4	.75690 970	40 416	012 6
.64	2.27176 025	45 422	473 5	.77467 578	42 897	777 6
.65	2.28349 668	+48 050	814 6	.79287 123	+45 596	279 6
.66	2.29571 405	50 914	455 8	.81152 309	48 540	934 3
.67	2.30844 107	54 057	715 6	.83066 085	51 753	499 7
.68	2.32170 922	57 504	236 8	.85031 671	55 277	109 4
.69	2.33555 305	61 298	490 7	.87052 598	59 148	790 0
.70	2.35001 059	+65 487	862 1	.89132 746	+63 414	559 2
.71	2.36512 384	70 132	431 1	.91276 392	68 137	222 4
.72	2.38093 936	75 290	600 6	.93488 270	73 375	017 1
.73	2.39750 888	81 044	755 6	.95773 633	79 210	289 1
.74	2.41489 012	87 492	187 5	.98138 334	85 740	437 3
.75	2.43314 776	+94 741	584 4	1.00588 924	+93 073	425 4

The limit of Table I, namely $\alpha = .75$, is also the limit of Runkle's and most other tables. Brown and Brouwer, however, tabulate to $\alpha = .845$ approximately, and also give *natural* values to 6 decimals for values of α at interval .001 between .90 and .95. It was decided to give b_0 in easily inter-

polable form up to $\alpha = .90$ in Table II, which has half the interval of Table I, and without provision for interpolation for larger values of α in Table III.

TABLE II

α	b_0	δ_m^2	δ^4	8-11
.750	2.43314 776	+ 23 703		584 4
.755	2.44262 761	24 698		103 1
.760	2.45235 455	25 753		471 4
.765	2.46233 915	26 883		537 7
.770	2.47259 271	28 082		103 1
.775	2.48312 725	+ 29 370		537 1
.780	2.49395 565	30 741		464 1
.785	2.50509 165	32 219		528 4
.790	2.51655 003	33 796		250 3
.795	2.52834 660	35 499		985 2
.800	2.54049 840	+ 37 330		002 4
.805	2.55302 377	39 307		700 9
.810	2.56594 250	41 442		983 7
.815	2.57927 598	43 758		816 8
.820	2.59304 740	46 269		002 6
.825	2.60728 192	+ 49 004		203 8
.830	2.62200 693	51 983		262 6
.835	2.63725 229	55 243		867 9
.840	2.65305 066	58 819		633 9
.845	2.66943 787	62 744		670 9
.850	2.68645 327	+ 67 079	+ 460	742 2
.855	2.70414 031	71 872	529	131 1
.860	2.72254 704	77 190	613	363 3
.865	2.74172 680	83 120	705	975 9
.870	2.76173 906	89 750	826	567 9
.875	2.78265 034	+ 97 203	+ 960	436 5
.880	2.80453 542	105 609	1 140	187 8
.885	2.82747 868	115 151	1 346	828 7
.890	2.85157 590	126 025	1 617	002 1
.895	2.87693 632	138 508	1 942	247 5
.900	2.90368 535	+ 152 911	+ 2 374	467 5

Table III has been given merely for convenience in case of further tabulation, and it is not recommended that it be interpolated, at any rate if many figures are required. When α is greater than .90, it is better either to compute b_0 directly as $2/M$, or to bring the calculation within the range of Table I or II by means of a single application of Landen's transformation, namely

$$b_0(\alpha) = (1 + \alpha_1)b_0(\alpha_1),$$

where

$$\alpha_1 = (1 - \alpha')/(1 + \alpha'), \quad \alpha' = \sqrt{1 - \alpha^2}.$$

TABLE III

α	b_0	α	b_0
·900	2·90368 53467 5	·950	3·29770 47204 6
·905	2·93196 78208 6	·955	3·35948 20865 7
·910	2·96195 22972 7	·960	3·42901 61223 3
·915	2·99383 64667 3	·965	3·50840 28131 1
·920	3·02785 42624 7	·970	3·60071 55105 2
·925	3·06428 52627 6	·975	3·71072 46450 5
·930	3·10346 73499 0	·980	3·84643 17671 7
·935	3·14581 40791 8	·985	4·02285 55766 0
·940	3·19183 90287 6	·990	4·27375 65222 2
·945	3·24219 07910 9	·995	4·70700 75468 5

The most striking illustration of this in the tables is

$$b_0(\cdot 80) = 1\cdot 25b_0(\cdot 25),$$

which may be verified to hold with the 11-decimal values given. Another useful illustration is

$$b_0(\cdot 96) = 1\cdot 5625b_0(\cdot 5625),$$

which may easily be checked to 8 decimal places, the interpolated value of $b_0(\cdot 5625)$ being 2·19457 032.

In Table II it needs to be noted that δ^4 becomes greater than 1000 at $\alpha = \cdot 88$, so that a few values of δ^4 are given, for convenience in applying the residual fourth-difference correction,*

$$+ T^{IV}(\delta^4 f_0 + \delta^4 f_1),$$

which is quite accurate enough for use with Everett modified second-difference interpolation in the present case. Since the maximum numerical value of the residual coefficient T^{IV} is only about 0·00023, it will be seen that even in the last interval of Table II the maximum residual correction amounts to barely a unit in the 8th decimal place of b_0 , and is consequently negligible for many purposes.

The last few modified differences in Table II, and also the last in Table IV, contain the term $+ 0\cdot 038\delta^6$. This throw-back takes quite sufficient account of the sixth differences, which are on the verge of being altogether negligible, and which need not be considered by anyone interpolating in the 8-decimal tables with the differences provided.

As α tends to unity, b_0 and b_1 tend to infinity, but $b_0 - b_1$ tends to $4/\pi$, and is much more easily interpolated. Thus it was decided to provide for the interpolation (up to ·97) of $b_0 - b_1$ rather than b_1 in Table IV, which gives both these quantities at interval ·01 in the range of α from ·75 to 1·00. It is hoped that this table may in most cases render unnecessary either the direct computation of b_1 or the use of the Landen-type formula, which is

$$b_1(\alpha) = \frac{1}{2}ab_0(\alpha) + \sqrt{a_1}b_1(a_1),$$

with the same notation as before.

* *Interpolation and Allied Tables*, pp. 800 (table of T^{IV}) and 928.

TABLE IV

α	b_1	$b_0 - b_1$	δ_m^2	8-11
·75	1·00588 92425 4	1·42725 852	+ 1 666	158 9
·76	1·03132 76247 7	1·42102 692	1 582	223 7
·77	1·05778 15848 0	1·41481 113	1 489	255 1
·78	1·08534 54152 3	1·40861 023	1 400	311 7
·79	1·11412 67085 7	1·40242 332	1 302	164 6
·80	1·14424 89746 0	1·39624 943	+ 1 204	256 4
·81	1·17585 49329 1	1·39008 757	1 100	654 6
·82	1·20911 07006 7	1·38393 670	992	996 0
·83	1·24421 11839 3	1·37779 574	878	423 3
·84	1·28138 71125 4	1·37166 355	758	508 6
·85	1·32091 43579 4	1·36553 892	+ 627	162 8
·86	1·36312 64837 1	1·35942 055	492	526 2
·87	1·40843 19733 7	1·35330 708	345	834 2
·88	1·45733 83935 8	1·34719 703	181	252 0
·89	1·51048 71336 5	1·34108 877	+ 6	665 6
·90	1·56870 48052 2	1·33498 054	- 188	415 3
·91	1·63308 19025 7	1·32887 039	405	947 0
·92	1·70509 81279 1	1·32275 613	652	345 7
·93	1·78683 20807 1	1·31663 527	941	691 9
·94	1·88133 41143 9	1·31050 491	1 274	143 7
·95	1·99334 30642 8	1·30436 160	- 1 688	561 8
·96	2·13081 47895 2	1·29820 133	2 199	328 1
·97	2·30869 68523 6	1·29201 866	- 2 905	581 6
·98	2·56062 52799 5	1·28580 649		872 2
·99	2·99420 24761 2	1·27955 405		461 0
1·00	∞	1·27323 954		447 4

3. The calculations for these tables have been made on a considerable variety of machines in the Mathematical Laboratory of the University of Liverpool. The author was responsible for the computation of the whole of Table I and parts of the other tables. He is greatly indebted to Professor L. Rosenhead, at whose instance the Joint Committee on Research of the university kindly provided a small grant, which enabled the skilful assistance of Miss Kathleen Mounfield, B.Sc., to be utilized in completing Tables II, III and IV, as well as in constructing the tables of elliptic integrals mentioned above. It is also a pleasure to acknowledge the great benefit which has accrued to the tables through the author's frequent opportunities for informal discussion with his colleague, Dr. J. C. P. Miller.

Department of Applied Mathematics,
University of Liverpool :
1938 December 2.

REMARKS ON THE EQUILIBRIUM OF DISTORTED POLYTROPES.

Zdeněk Kopal, D.Sc., Research Associate, Harvard College Observatory.

The equations of the mechanical equilibrium of tidally distorted gaseous configurations are, as is well known,

$$\text{grad } P = \rho \text{ grad } (V + V_T), \quad (1)$$

where P is the pressure, ρ the density, V and V_T the potential arising from the mass of the configuration and the tide-generating potential respectively, which must satisfy the following equations:—

$$\left. \begin{aligned} \nabla^2 V &= -4\pi G\rho, \\ \nabla^2 V_T &= 0, \end{aligned} \right\} \quad (2)$$

G being the constant of gravitation. From (1) and (2) it follows that

$$\text{div} \left(\frac{1}{\rho} \text{grad } P \right) = -4\pi G\rho. \quad (3)$$

Changing over to polar co-ordinates, neglecting the longitude on account of the spheroidal symmetry and introducing the polytropic variables

$$\begin{aligned} r &= a\xi, \\ \rho &= \lambda \Theta^n, \end{aligned}$$

where

$$a^2 = \frac{(n+1)K}{4\pi G} \lambda^{\frac{1}{n}-1},$$

K and n are defined by

$$P = K\rho^{1+\frac{1}{n}},$$

and λ is the value of the central density, eqn. (3) reduces to

$$\nabla^2 \Theta = -\Theta^n, \quad (4)$$

a fundamental equation in the recent investigations of Chandrasekhar.*

The aim of the present paper is first to show that in the special case $n = 1$, eqn. (4) admits of a more general solution than that given by Chandrasekhar, and, as will be seen, that all functions involved in the new solution can be expressed in a finite number of terms. This will enable one, in the special case $n = 1$ at least, to avoid the long and tedious integration in series, as done by Chandrasekhar.

Assume the normal solution of (4) to be of the form

$$\Theta = \psi(\xi)\phi(\mu), \quad (5)$$

* *M.N.*, 93, 390, 449, 462, 539, 1933.

where $\psi(\xi)$ is a purely radial function and $\phi(\mu)$ depends only on μ (*i.e.* on the cosine of the latitude). Inserting (5) in (4) we find that (4) can be written as a linear combination of the following two equations:—

$$\frac{d}{d\xi}\left(\xi^2 \frac{d\psi}{d\xi}\right) + (\xi^2 - m)\psi = 0, \quad (6)$$

$$\frac{d}{d\mu}\left((1 - \mu^2) \frac{d\phi}{d\mu}\right) + m\phi = 0, \quad (7)$$

where m is an arbitrary constant. Putting

$$m = j(j+1),$$

j being an integer, we see that (7) reduces to the well-known Legendre equation, and the general solution of (4) can consequently be put in the form:

$$\Theta = \sum_{j=0}^{\infty} a_j \psi_j(\xi) P_j(\mu), \quad (8)$$

$P_j(\mu)$ being Legendre polynomials of different orders and the a_j 's constants to be determined later.

The solution of eqn. (6) can most conveniently be expressed in terms of Bessel functions.* Remembering that Θ is to be unity for $\xi = 0$, we find

$$\psi_j(\xi) = 2^j \sqrt{\frac{2}{\xi}} \Pi\left(j + \frac{1}{2}\right) \mathcal{Y}_{j+\frac{1}{2}}(\xi). \quad (9)$$

This solution holds good for all values of j . But in our case the j 's happen to be positive integers, and the Bessel functions of the order half an odd integer admit an expression in finite number of terms. It can be proved † that

$$\mathcal{Y}_{j+\frac{1}{2}}(\xi) = (-1)^j \xi^{j+\frac{1}{2}} \left(\frac{d}{\xi d\xi}\right)^j \left(\frac{\sin \xi}{\xi}\right), \quad (10)$$

* Another method of solution consists in substituting

$$\psi_j = \frac{u_j}{\xi},$$

which reduces (6) to

$$\frac{d^2 u_j}{d\xi^2} + u_j = \frac{j(j+1)}{\xi^2} u_j,$$

the solution of which is

$$u_j = a_j \xi^{-j} \left(\frac{d}{dr}\right)^j \left(\frac{\cos \xi \sqrt{r}}{\sqrt{r}}\right),$$

where r is to be put equal to unity after the performance of the differentiations (*cf.* Forsyth, *Treatise on Differential Equations*, London, p. 196, 1903)—a solution obviously identical with that given above. It may also be noted that (6) admits of an integral representation, being satisfied by

$$\psi_j(\xi) = \frac{\xi^j}{\sqrt{\pi}} \cdot \frac{\Pi(j+\frac{1}{2})}{j!} \int_{-1}^{+1} e^{i\xi t} (1-t^2) dt.$$

† *Cf.* Whittaker and Watson, *Modern Analysis*, Cambridge, pp. 364–365, 1924; or Poole, *Theory of Linear Differential Equations*, Oxford, pp. 147–151, 1936.

or we can start with the well-known expressions

$$\mathcal{Y}_{\frac{1}{2}}(\xi) = \sqrt{\left(\frac{2}{\pi\xi}\right)} \sin \xi, \quad \mathcal{Y}_{-\frac{1}{2}}(\xi) = \sqrt{\left(\frac{2}{\pi\xi}\right)} \cos \xi$$

and evaluate $\mathcal{Y}_{j+\frac{1}{2}}(\xi)$ using the recurrence formula

$$\mathcal{Y}_{n+1}(\xi) = \frac{2n}{\xi} \mathcal{Y}_n(\xi) - \mathcal{Y}_{n-1}(\xi).$$

Combining (9) and (10) and performing the differentiations the $\psi_j(\xi)$ take the forms :

$$\psi_0(\xi) = \frac{\sin \xi}{\xi},$$

$$\psi_1(\xi) = 3 \left\{ \frac{\sin \xi}{\xi^2} - \frac{\cos \xi}{\xi} \right\},$$

$$\psi_2(\xi) = 15 \left\{ \left(\frac{3}{\xi^2} - 1 \right) \frac{\sin \xi}{\xi} - \frac{3 \cos \xi}{\xi} \right\},$$

$$\psi_3(\xi) = 105 \left\{ \left(\frac{15}{\xi^3} - \frac{6}{\xi} \right) \frac{\sin \xi}{\xi} - \left(\frac{15}{\xi^2} - 1 \right) \frac{\cos \xi}{\xi} \right\},$$

$$\psi_4(\xi) = 945 \left\{ \left(\frac{105}{\xi^4} - \frac{45}{\xi^2} + 1 \right) \frac{\sin \xi}{\xi} - \left(\frac{105}{\xi^3} - \frac{10}{\xi} \right) \frac{\cos \xi}{\xi} \right\},$$

$$\psi_5(\xi) = 10395 \left\{ \left(\frac{945}{\xi^5} - \frac{420}{\xi^3} + \frac{15}{\xi} \right) \frac{\sin \xi}{\xi} - \left(\frac{945}{\xi^4} - \frac{105}{\xi^2} + 1 \right) \frac{\cos \xi}{\xi} \right\},$$

$$\psi_j(\xi) = 1 \cdot 3 \cdot 5 \cdot \dots \cdot (2j+1) \{ U_n(\xi) \sin \left(\xi - \frac{1}{2}n\pi - \frac{1}{4}\pi \right) + V_n(\xi) \cos \left(\xi - \frac{1}{2}n\pi - \frac{1}{4}\pi \right) \}, \quad (11)$$

where

$$U_n(\xi) = \sum_{i=1}^{\infty} \frac{(-1)^i (4n^2 - 1^2)(4n^2 - 3^2) \cdot \dots \{4n^2 - (4i-3)^2\}}{(2i-1)! 2^{6i-3} \xi^{2i}}$$

$$V_n(\xi) = \frac{1}{\xi} + \sum_{i=1}^{\infty} \frac{(-1)^i (4n^2 - 1^2)(4n^2 - 3^2) \cdot \dots \{4n^2 - (4n-1)^2\}}{(2i)! 2^{6i} \xi^{2i}}.$$

$$n = j + \frac{1}{2}.$$

The derivatives of $\psi_j(\xi)$ can again be easily deduced from (11) using the recurrence formula

$$2\mathcal{Y}_n'(\xi) = \mathcal{Y}_{n-1}(\xi) - \mathcal{Y}_{n+1}(\xi),$$

and because they occur explicitly in the determinations of a_j 's their values may be given here :

$$\psi_0'(\xi) = \left\{ \frac{\cos \xi}{\xi} - \frac{1}{\xi} \frac{\sin \xi}{\xi} \right\},$$

$$\psi_1'(\xi) = 3 \left\{ \frac{6 \cos \xi}{\xi} - \left(\frac{6}{\xi^2} - 3 \right) \frac{\sin \xi}{\xi} \right\},$$

$$\begin{aligned}
 \psi_2'(\xi) &= 15 \left\{ \left(\frac{9}{\xi^2} - 1 \right) \frac{\cos \xi}{\xi} - \left(\frac{9}{\xi^3} - \frac{4}{\xi} \right) \frac{\sin \xi}{\xi} \right\}, \\
 \psi_3'(\xi) &= 105 \left\{ \left(\frac{60}{\xi^3} - \frac{7}{\xi} \right) \frac{\cos \xi}{\xi} - \left(\frac{60}{\xi^4} - \frac{27}{\xi^2} + 1 \right) \frac{\sin \xi}{\xi} \right\}, \\
 \psi_4'(\xi) &= 945 \left\{ \left(\frac{525}{\xi^4} - \frac{65}{\xi^2} + 1 \right) \frac{\cos \xi}{\xi} - \left(\frac{525}{\xi^5} - \frac{240}{\xi^3} + \frac{11}{\xi} \right) \frac{\sin \xi}{\xi} \right\}, \\
 \psi_5'(\xi) &= 10395 \left\{ \left(\frac{5670}{\xi^5} - \frac{735}{\xi^3} + \frac{16}{\xi} \right) \frac{\cos \xi}{\xi} - \left(\frac{5670}{\xi^6} - \frac{2625}{\xi^4} + \frac{135}{\xi^2} - 1 \right) \frac{\sin \xi}{\xi} \right\}. \quad (12)
 \end{aligned}$$

A glance at (11) shows that for $j=0$, $\psi_0(\xi)$ becomes the solution of the Emden equation in the non-rotating state, and in view of our system of units $a_0=1$. The a_j 's for $j>0$ must obviously depend on the distance and mass of the disturbing body, and their values can be obtained in comparing the inner and outer potential at the boundary, where these must be continuous. If the distortions are slight, the comparison can be made at the surface of the initial sphere of the radius $\xi=\xi_1$, ξ_1 being the first zero solution of the corresponding Emden equation and equal to π . As in this procedure the numerical values of $\psi_j(\xi_1)$ and $\psi_j'(\xi_1)$ are involved,* they have been calculated with the aid of (11) and (12) correctly to seven significant figures and the results are :

$\psi_0(\xi_1) =$	0.0000000	$\psi_0'(\xi_1) =$	- 0.3183102
$\psi_1(\xi_1) = +$	0.9549306	$\psi_1'(\xi_1) =$	- 0.1823780
$\psi_2(\xi_1) = +$	4.559450	$\psi_2'(\xi_1) = +$	0.4206975
$\psi_3(\xi_1) = +$	17.37356	$\psi_3'(\xi_1) = +$	9.795356
$\psi_4(\xi_1) = +$	61.15624	$\psi_4'(\xi_1) = +$	59.02848
$\psi_5(\xi_1) = +$	207.2140	$\psi_5'(\xi_1) = +$	276.1952

So far we have dealt only with the tidal problem, but it can be briefly shown that a similar method can be applied to the rotational problem as well. The equations of the mechanical equilibrium for rotating configurations are :

$$\text{grad } P = \rho \text{ grad } (V + \Omega), \quad (13)$$

$$\left. \begin{aligned} \nabla^2 V &= -4\pi G\rho, \\ \nabla^2 \Omega &= 2\omega^2. \end{aligned} \right\} \quad (14)$$

Ω being the centrifugal potential and ω the angular velocity of rotation. From (13) and (14) it follows again that

$$\text{div} \left(\frac{1}{\rho} \text{grad } P \right) = -4\pi G\rho + 2\omega^2, \quad (15)$$

* It is perhaps unnecessary to calculate here the whole march of the $\psi_j(\xi)$'s within the interval $0 < \xi < \pi$, because their values can easily be found using the tables for Bessel functions of half an odd integer order tabulated, e.g. in *British Association Report*, 1916.

and on going through exactly the same procedure as before we get

$$\left. \begin{aligned} \nabla^2 \Theta &= -\Theta + v, \\ v &= \frac{\omega^2}{2\pi G \rho} \end{aligned} \right\} \quad (16)$$

Substituting

$$\Theta = \psi(\xi)\phi(\mu) + v, \quad (17)$$

we find the functions $\psi(\xi)$ and $\phi(\mu)$ again to be defined by eqns. (6) and (7) and the general solution again to be (8), but the condition $\Theta = 1$ for $\xi = 0$, no matter what v , gives $a_0' = (1 - v)$. Hence the solution (8) can be rewritten in the form

$$\Theta = \psi_0(\xi) + v(1 - \psi_0(\xi)) + \sum_{j=1}^{\infty} a_j' \psi_j(\xi) P_j(\mu), \quad (18)$$

which is identical in fact with that of Chandrasekhar.* But the $\psi_j(\xi)$'s are now expressed in a finite number of terms. The determination of a_j 's proceeds again by comparing the inner and outer potential at $\xi = \xi_1$, and the numerical values of $\psi_j(\xi_1)$ and $\psi_j'(\xi_1)$ given above may again be useful.† It is only obvious at once that for $v = 0$ and $j > 0$ also $a_j' = 0$.

General Discussion and Summary.—The aim of the present paper has been, first, to show that the fundamental equations (4) and (16) governing the density distribution of a rotationally or tidally (or both) distorted configuration with initially polytropic structure specified by $n = 1$ admit normal solutions of the form

$$\sum_j a_j \psi_j(\xi) \phi_j(\xi).$$

The radial functions for $j = 2$ are identical with those which Chandrasekhar denoted as “associated Emden functions” and defined by infinite series. In the present paper these functions have been generalized down to $j = 0$ and expressed in a finite number of terms.

It is well known that within the interval $5 > n > 0$ the value $n = 1$ is the only one which enables one to solve the Emden equation

$$\frac{1}{\xi^2} \frac{d}{d\xi} \left(\xi^2 \frac{d\psi_0}{d\xi} \right) = -\psi_0,$$

in finite terms, the solution being

$$\psi_0 = \frac{\sin \xi}{\xi}.$$

* *M.N.*, 93, 392, eqn. (15).

† See *M.N.*, 93, p. 398, eqn. (53), and p. 456, eqns. (42) and (43). The comparison of the $\psi_j(\xi_1)$ and $\psi_j'(\xi_1)$ as evaluated from the strict formulæ given in the present paper and the expansions used by Chandrasekhar (see *loc. cit.*, p. 456, Table VI) can give us some idea of the accuracy of Chandrasekhar's numerical integration. His results are found to be correct nearly to four significant figures.

In the present paper we have proved that the associated Emden equation for $n = 1$,

$$\frac{1}{\xi^2} \frac{d}{d\xi} \left(\xi^2 \frac{d\psi_j}{d\xi} \right) = - \left(1 - \frac{j(j+1)}{\xi^2} \right) \psi_j,$$

is also soluble in a finite number of terms for every integral value of j , the associated Emden functions assuming the form

$$\psi_j(\xi) = U_j \left(\frac{1}{\xi} \right) \frac{\sin \xi}{\xi} + V_j \left(\frac{1}{\xi} \right) \frac{\cos \xi}{\xi}.$$

The function $\frac{\sin \xi}{\xi}$ may consequently be regarded as an associated Emden

function of the order zero. Apart from the case $n = 1$, however, no other associated Emden function of any order can be expressed in finite terms, and recursion to infinite series is inevitable.

In conclusion, I should like to record my indebtedness to Dr. Harlow Shapley for the facilities offered by the Harvard Observatory, where the above work was carried out.

*Harvard College Observatory,
Cambridge, Mass. :
1938 November.*

THE FISSION OF ROTATING BODIES.

Lieut.-Col. K. E. Edgeworth, D.S.O., M.C.

M.N., 98, 8, 1938, contains two papers by Dr. R. A. Lyttleton dealing, amongst other matters, with the fission of a rotating body which has reached a condition of instability. The starting-point is the limiting ellipsoid of Jacobi with its axes in the ratios

$$a : b : c : (abc)^{\frac{1}{3}} = 1.866 : 0.815 : 0.651 : 1.$$

It has commonly been accepted that the rotating body, on reaching this stage, divides into two unequal parts which continue to revolve about each other in approximately circular orbits, but the intermediate stages in this process of development are by no means clear. Dr. Lyttleton suggests therefore, as an alternative solution, that the energy of rotation is transformed in such a way that the two bodies are driven apart with sufficient velocity to escape entirely from each other's influence ; and it is shown that, in appropriate conditions, the rotational energy available is sufficient for the purpose.

Now any valid solution to a problem of this character must evidently be capable of interpretation in terms of positions and velocities, and it is not a difficult matter to draw a diagram which will give a rough idea of what is alleged to have taken place. Referred to their common centre of gravity the orbits of the two bodies are, of course, similar in character, and it is sufficient to discuss the movements of the smaller body or satellite. The data are :

(i) At the beginning of the disturbance which ends in separation the material, which is to form the satellite, is revolving in a circular orbit whose radius is somewhat less than a , say $.8a$.

(ii) When the bodies have actually separated the only significant force between them is their gravitational attraction, and any special forces which may have come into play in the act of separation must be assumed to have ceased. It seems clear that any effective contact between the bodies must have ceased by the time that the distance between them has reached, say, $1.3a$, and it follows that no significant force, other than gravitational attraction, can be postulated beyond that distance.

(iii) Any gradual process of evolution raises difficulties of its own, and is specifically rejected in the solution which we are discussing ; the solution assumes a rapid and continuous transition from a circular orbit to a parabolic orbit, so that the satellite must follow some such path as that indicated in fig. 1, and must arrive at B with considerably more kinetic energy than it originally possessed at A. In its passage from A to B, therefore, the satellite must have been acted on by some unknown force appreciably greater than

the force of gravity. The approximate direction of this force on the diagram would be vertically downwards, as indicated by the arrow.

(iv) The time taken for the satellite to traverse the distance from A to B can be roughly estimated; it is less than one-third of the original period of revolution. The energy acquired by the satellite is supposed to have been derived from the rotational and/or the potential energy of the primary, and the theory requires that the speed of rotation and/or the shape of the primary

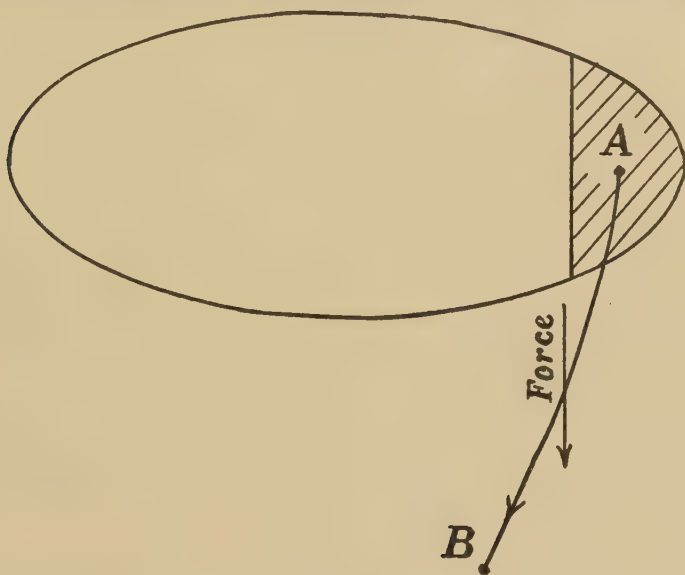


FIG. 1.

should be so altered as to release the necessary energy within this very brief period. Simultaneously the energy so released must be transferred to the satellite.

It is abundantly clear that there is no known mechanism which would explain these large and rapid transfers of energy from one part of the system to another, and to put the matter quite simply the suggested solution (unless I have misunderstood it) is a mechanical absurdity.

Let us then return to the original problem. It is no doubt true that the course of events which must succeed the appearance of the limiting Jacobian ellipsoid cannot be precisely determined, but we are not thereby debarred from discussing the matter in general terms.

It is known, of course, that there exists a stable configuration in which two fragments of the original body revolve about each other in circular or nearly circular orbits, but this configuration is only possible when the distance between the two bodies exceeds the limit—approximately $1.3a$ —discussed by Darwin and Roche, and the question at issue is whether transition from the original stable state represented by the limiting Jacobian ellipsoid to this second stable state is dynamically possible; and, supposing that it is dynamically possible, what are the intermediate stages? In particular, is it possible

for a body of appreciable size to cross the danger zone and reappear in safety outside Roche's limit?

The problem is briefly discussed by Sir James Jeans (*Cosmogony and Stellar Dynamics*, p. 287, 1929), who suggests that the original rotating body assumes a distorted pear-shaped form in which the pointed end of the pear lags behind the general rotation, so that the body assumes some such shape as that indicated in fig. 2. In this stage it is clear that the pointed end will be accelerated by the local gravitational field and will be driven outwards, so

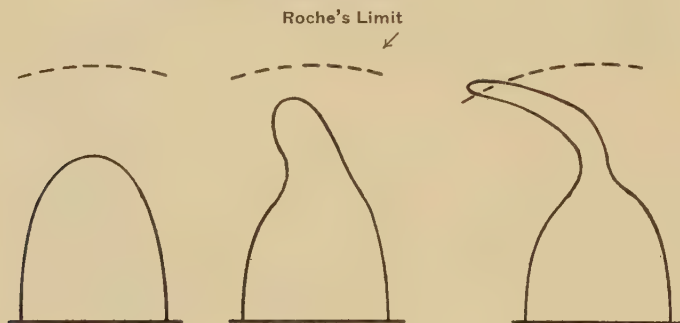


FIG. 2.

that a further stage would be reached which might appropriately be described as an expanding one-armed spiral. Alternatively, the appearance of a two-armed spiral is by no means an impossibility.

In either case it is clear that a considerable portion of the detached material must remain inside Roche's limit, and must ultimately disintegrate into a swarm of small particles resembling Saturn's rings; but, at the same time, it is not unreasonable to argue that the tip of the spiral arm might escape and might be preserved intact to form a satellite.

The fate of the disintegrated material is also of importance. It is clear that any forces which tend to disturb the orbital motion of the particles involve the transfer of energy and angular momentum; at the same time they cause collisions between the particles, which in turn involve loss of energy. In the case of Saturn's rings the planet itself is spheroidal in shape (or nearly so), and disturbances in the orbital motions of the particles arise chiefly from the action of external bodies such as the Sun and the satellites; in that case the particles lose angular momentum and fall inwards towards the planet. In the case we are now considering, on the other hand, the chief cause of disturbance is the ellipsoidal shape of the larger mass, and in that case the particles gain angular momentum and are driven outwards.

It thus appears that the detached material will be driven away from the larger body, so that it is likely to be absorbed by the satellite, always assuming that there exists an embryo satellite to act as a nucleus for the process of condensation.

The various stages through which a rotating body seems likely to pass are therefore :

- (i) An oblate spheroid, becoming more oblate and changing into
- (ii) An ellipsoid, which gradually becomes more elongated and changes into
- (iii) A distorted pear-shaped body, the pointed end of which becomes rapidly longer and more inclined until it can be described as
- (iv) An expanding one-armed spiral. In due course the tip of the spiral escapes and the intermediate material disintegrates, forming
- (v) A satellite and a swarm of disintegrated material resembling Saturn's rings. The bulk of the material forming these rings is driven outwards and is eventually captured by the satellite, so that the final result is
- (vi) Two separate bodies revolving about each other in nearly circular orbits, but still subject to mutual interference in the form of tidal friction.

Summary.—The paper is concerned with Dr. Lyttleton's theory that a rotating ellipsoid may divide into two parts and that the process of fission may be of such a character that the two fragments separate from one another at hyperbolic velocities. It is pointed out that such a process would involve for a certain time a repulsive force between the two fragments exceeding the force of gravity, and that there is no known mechanism by which repulsive forces of this magnitude could be produced. Reference is then made to the well-known theory that fission might lead to the production of a satellite, and it is pointed out that scattered material surrounding a rotating ellipsoid would be driven outwards and might subsequently condense outside Roche's limit to form a satellite.

OCCULTATIONS OF STARS BY THE MOON OBSERVED AT MELBOURNE OBSERVATORY IN 1937.

J. M. Baldwin.

(Communicated by D. H. Sadler)

The observations of the occultations in this report were made by the eye and ear method throughout, the clock beating on a relay in the observing room. The telescope generally used was an 8-inch refractor, but occasionally when this was in use for other purposes, the 10-inch guiding telescope of the astrograph was used for occultations.

The reductions were computed using the method and reduction elements given in the *Nautical Almanac* for 1938; the computations were carried out in duplicate.

TABLE I

No.	Date	Star	Mag.	Phase	U.T.		
					h	m	s
	1937						
1	Mar. 19	B.D. +23° 1087	7.0	D	9	42	21.62
2	20	B.D. +21° 1428	6.8	D	11	40	18.56
3	25	ρ^5 Leonis	5.4	D	13	54	53.24
4	Apr. 2	B.D. -22° 4423	6.2	R	16	09	10.54
5	3	121 B. Sagittarii	5.9	R	15	40	02.10
6	3	128 B. Sagittarii	6.4	R	19	34	29.52
7	4	267 B. Sagittarii	5.9	R	19	29	35.79
8	17	B.D. +19° 1784	6.8	D	12	09	27.76
9	May 17	B.D. +8° 2285	7.1	D	8	49	49.04
10	17	B.D. +8° 2289	6.7	D	10	23	36.85
11	June 15	ρ^5 Leonis	5.4	D	8	24	56.66
12	16	B.D. -5° 3416	6.8	D	12	26	05.70
13	25	267 B. Sagittarii	5.9	R	14	34	42.44
14	27	19 Aquarii	5.8	R	20	18	39.34
15	28	138 B. Aquarii	6.4	R	19	51	24.99
16	July 13	B.D. -2° 3411	7.5	D	10	29	16.73
17	20	B.D. -21° 4659	6.6	D	12	51	33.15
18	Aug. 15	B.D. -21° 4341	7.1	D	9	14	57.61
19	17	30 G. Sagittarii	6.2	D	10	57	12.71
20	17	B.D. -21° 4866	6.6	D	11	53	06.03
21	18	190 B. Sagittarii	5.4	D	16	30	26.03
22	Sept. 14	B.D. -20° 5223	6.8	D	11	32	36.56
23	16	β Capricorni	3.2	D	10	17	18.53
24	18	30 Aquarii	5.6	D	10	34	28.26
25	Nov. 8	B.D. -19° 5242	7.0	D	11	38	08.01
26	Dec. 12	δ Piscium	5.6	D	10	07	20.68

TABLE II

No.	p	q	p^2	pq	q^2	$\Delta\sigma$	$p.\Delta\sigma$	$q.\Delta\sigma$
1	+ 87	-49	76	-43	24	0.0	0.0	0.0
2	+100	+ 5	100	+ 5	0	-1.0	-1.0	0.0
3	+ 92	-39	85	-36	15	-1.0	-0.9	+0.4
4	- 75	-66	56	+50	44	+1.2	-0.9	-0.8
5	- 97	-23	95	+22	5	+1.8	-1.7	-0.4
6	- 76	-65	58	+49	42	+1.4	-1.1	-0.9
7	- 87	+49	76	-43	24	+2.0	-1.7	+1.0
8	+ 92	-40	84	-37	16	+0.6	+0.6	-0.2
9	+100	- 6	100	- 6	0	-0.4	-0.4	0.0
10	+ 92	-39	85	-36	15	+0.2	+0.2	-0.1
11	+ 81	+58	66	+47	34	-1.9	-1.5	-1.1
12	+100	- 9	99	- 9	1	-0.6	-0.6	+0.1
13	- 85	-53	72	+45	28	+0.8	-0.7	-0.4
14	- 53	-85	28	+45	72	+1.8	-1.0	-1.5
15	- 62	+78	39	-49	61	+0.6	-0.4	+0.5
16	+ 99	+16	97	+16	3	-1.6	-1.6	-0.3
17	+ 44	+90	19	+40	81	-1.3	-0.6	-1.2
18	+ 99	-11	99	-11	1	-1.4	-1.4	+0.2
19	+ 82	+58	67	+47	33	-1.9	-1.6	-1.1
20	+ 95	+31	90	+29	10	-1.8	-1.7	-0.6
21	+ 89	-45	80	-40	20	-2.6	-2.3	+1.2
22	+ 23	+97	5	+22	95	-0.7	-0.2	-0.7
23	+ 53	+85	28	+45	72	-1.4	-0.7	-1.2
24	+ 54	-84	29	-45	71	+0.8	+0.4	-0.7
25	+ 82	-58	67	-47	33	+0.2	+0.2	-0.1
26	+ 46	+89	21	+41	79	-1.0	-0.5	-0.9

Erratum

M.N., 99, 50, R. Obs., Greenwich, Stellar Parallaxes, No. 10, Br. 1446,

for Proper Motion -35 read -30.

for Parallax +11 read +23.

MICROMETRICAL MEASURES OF DOUBLE STARS. (THIRTIETH SERIES.)

W. Milburn.

The micrometrical measures are given as in previous lists and were all made with the 24-inch (Calver) reflector.

Name	B.D.	1900		P.A.	Dist.	Mags.	Nights	Date 1938 +	
		R.A.	Decl.						
		^h ^m	[°] [']	[°]	"				
Σ 227	+29.371	2 6.6	+29 50	69.2	4.14	...	1	.890	
OS 433	+34.4371	21 13.8	34 29	221.6	...	...	1	.832	AB
				179.6	...	...	1	.832	AC
Es 520	+27.4125	34.5	27 59	38.4	7.41	...	1	.909	
Es 521	+28.4171	39.9	28 19	276.5	16.24	...	1	.909	
Es 394	+29.4764	22 42.2	30 4	341.3	3.67	...	2	.899	AB
				288.7	80 ±	...	1	.909	AC
M 586	...	46.3	29 30	12.0	6.48	...	1	.890	
Es 399	+29.4937	23 24.3	29 17	203.8	7.32	9.5, 12.0	2	.901	
Es 400	+29.4938	24.6	29 18	214.8	5.98	9.5, 10.5	2	.901	

Missing B.D. Star

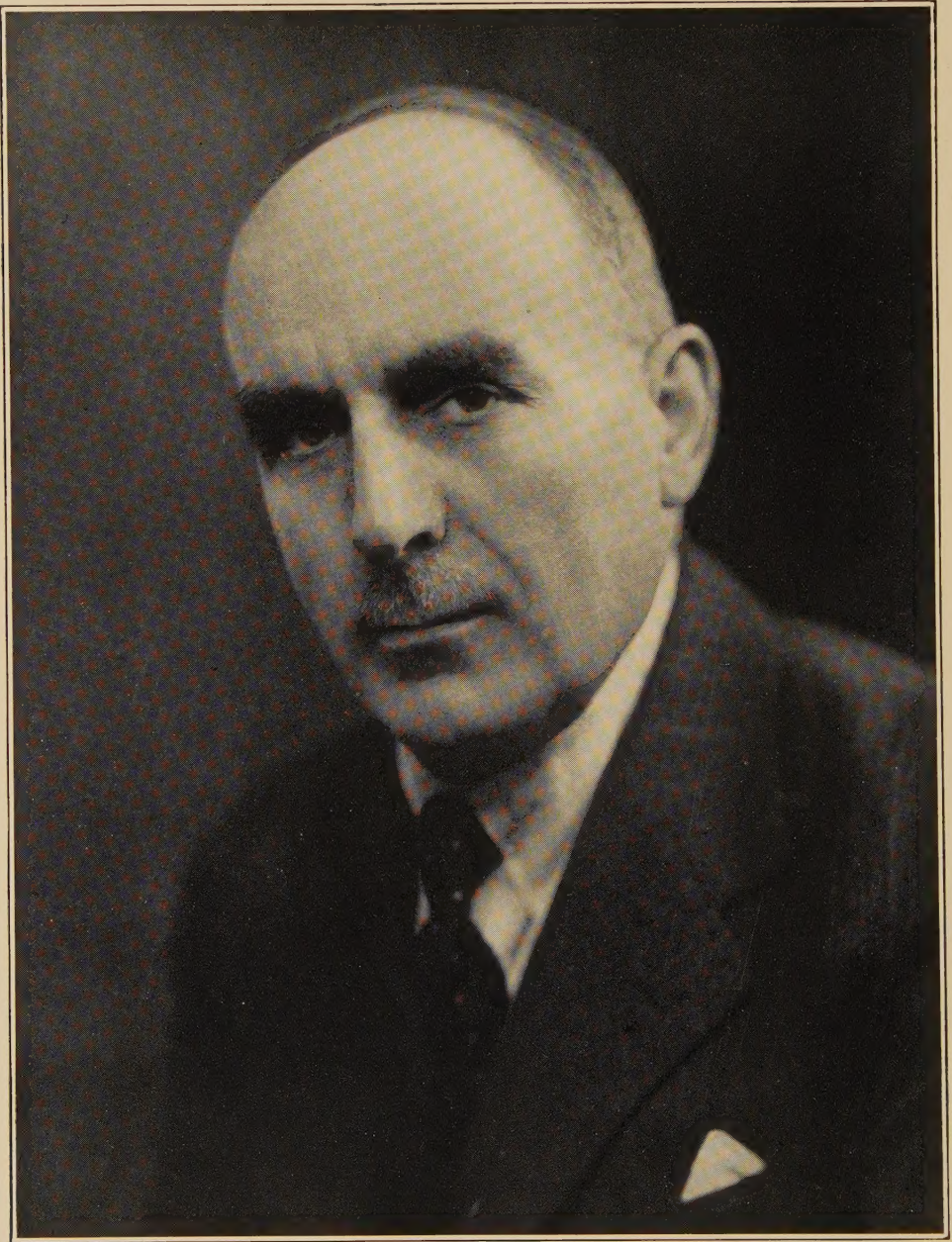
B.D. +34°.1546 : 7^h 1^m 4^s.1 ; +34° 10'.4 (1855) : mag. 9.5 ; 1938 March 16, 22 ; missing.

There are three stars south of the catalogue place : the nearest is 4'.3 S.

New Double Stars

Milburn No.	B.D.	1900		P.A.	Dist.	Mags.	Nights	Date 1938 +	
		R.A.	Decl.						
		^h ^m	[°] [']	[°]	"				
1033	+29.235	1 20.5	+29 22	102.4	65 ±	9.5, 11.0	1	.890	AB
				61.4	3.46	C = 11.2	1	.890	BC
1034	...	2 5.7	30 34	285.8	4.75	9.6, 12.0	1	.055	
1035	+33.554	53.8	33 16	277.9	17.29	9.5, 11.0	1	.011	AB
				222.4	37 ±	C = 11.0	1	.011	AC
				145.3	5.74	D = 11.5	1	.011	CD
1036	...	4 18.1	34 7	217.6	5.31	11.0, 11.5	1	.011	
1037	+34.1023	5 16.6	34 3	278.6	3.98	9.0, 11.0	2	.134	
1038	...	20.0	34 13	93.3	3.21	9.6, 10.3	1	.178	
1039	...	21.2	34 41	113.3	6.76	10.5, 10.7	2	.127	
1040	...	21.7	34 42	192.6	4.50	10.0, 11.2	2	.059	
1041	+34.1235	54.4	34 6	174.6	8.09	9.5, 10.2	2	.155	
1042	...	56.5	34 38	255.5	3.50	9.6, 12.3	3	.117	
1043	...	56.7	34 40	270.8	2.71	11.0, 13.3	3	.117	
1044	...	6 14.8	34 5	123.6	6.57	9.6, 10.0	2	.155	
1045	+34.1330	16.6	34 21	152.3	6.11	9.5, 10.0	2	.175	
1046	...	18.5	33 57	356.8	3.86	11.5, 11.5	2	.175	
1047	...	22.6	34 43	85.7	4.97	9.9, 10.1	2	.211	
1048	...	7 4.7	33 54	36.2	5.83	9.5, 10.0	2	.196	
1049	+34.1760	8 3.8	34 4	293.7	5.50	9.4, 9.7	1	.178	
1050	+26.4259	21 41.1	26 41	64.3	7.28	9.5, 10.5	2	.869	
1051	...	22 9.4	25 30	97.3	2.16	11.0, 11.2	1	.906	

*The Observatory, Tow Law,
Co. Durham.*



Dr. H. Spencer Jones, F.R.S., *Astronomer Royal,*
President, 1937-39.